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PLANE TRIGONOMETRY,

AND

LOGARITHMS.

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LOGARITHMS.

FOR SCHOOLS AND PRIVATE STUDENTS.

BY

JOHN WALMSLEY,

MATHEMATICAL MASTER OF ABINGTON HOUSE SCHOOL, NORTHAMPTON.

LONDON:

C. F. HODGSON AND SON,

GOUGH SQUARE, FLEET STREET.

1865.

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PREFACE.

It has been attempted in this work to give a thoroughly modern account of those portions of Plane Trigonometry, which an intelligent student may be expected to read as his first course of the subject. In the execution of it, two principal aims have been kept in view.

In the first place, it has been sought to preserve clearness throughout—clearness of expression, and of demonstration. But to accomplish this, it has not been thought advisable to banish all detail in the text, and to give outline only, except in cases where the student could not fairly expect more. An *air* of greater simplicity might perhaps thus have been given to the book; but that, of course, would soon have vanished on its being read. On the other hand, it is hoped that it will seldom, if ever, be found to occur, that a multitude of words has been allowed to render a difficulty in the nature of the subject still more difficult.

The second main endeavour has been to place, group,

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EXERCISE I.—1. Write in their simplest forms the ratios of 1 to $\frac{1}{2}$, $\frac{1}{3}$ to $\frac{3}{4}$, $\frac{3}{4}$ to $3\frac{1}{4}$, 7 to 21, 15 to 85.

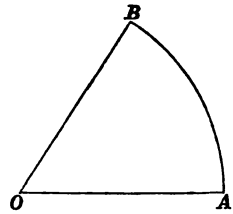
2. What are the ratios of a foot to a yard, a yard to $10\frac{1}{2}$ inches, $1\frac{1}{3}$ feet to $1\frac{1}{2}$ inches, 6 furlongs to $1\frac{1}{3}$ miles?

3. Compare the ratio of a foot to a yard with the ratio of a square foot to a square yard.

4. Compare the ratio of a foot to an inch with that of a square foot to an area of 3 inches by 4 inches.

6. *Unit of Circular Measure.*—Besides the right angle, there is another angle which has a fixed magnitude, and is of very great convenience for measuring angles.

Take any point as centre, O for instance, and with any straight line as radius, as OA, describe an arc of a circle, AB suppose; make the arc AB equal in length to OA, and join OB. The angle AOB can be proved to be of the same magnitude for all lengths of OA, and consequently of AB. AOB is the angle.

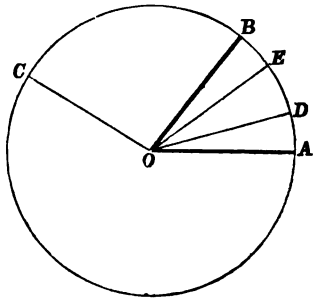


By reference to this angle also we measure the magnitudes of angles generally, and form a method distinct from those previously mentioned. We do not here, as in the case of the right angle, divide and subdivide into portions, and give names to each portion; but every angle measured is expressed with direct reference to this angle itself, and in terms of it. It is itself called the unit of circular measure; and therefore an angle twice as large is 2, another half as large is $\frac{1}{2}$; and so on. Formally defining, we have

The unit of circular measure is the angle at the centre of any circle subtended by an arc equal to the radius.

7. *Circular Measure.*—For the application of this method, it is necessary to understand clearly the following proposition, which will be here stated and explained, but not proved:

An angle at the centre of a circle is measured in terms of the unit of circular measure by the ratio of the arc which subtends it to the radius of the circle.



For instance, in any circle ABC, in which arc AB = radius OA, and therefore $\angle AOB = 1$, if $\angle AOC = 3\angle AOB = 3$, then it will be found also that the ratio

$$\frac{\text{arc } AC}{OA} = 3;$$

which ratio therefore expresses the angle. Also, if

$$\angle AOD = \frac{1}{4}\angle AOB = \frac{1}{4};$$

then it will be found that

$$\frac{\text{arc } AD}{OA} = \frac{1}{4};$$

and so for all angles.

All angles are included in this proposition; for any angle whatever may be made to have its position at the centre of a circle; and it is implied in it, that whatever be the radius of the circle described about that centre, the ratio mentioned is the same for the same angle.

That which measures an angle in terms of the unit of circular measure is called the *circular measure* of it. From the proposition above it follows that

The circular measure of any angle at the centre of a circle is the ratio of the subtending arc to the radius of the circle.

Wherefore in the circle ABC, if arc $AC = 3OA$, then in circular measure

$$\angle AOC = \frac{AC}{OA} = \frac{3OA}{OA} = 3;$$

and similarly, if arc $AE = \frac{3}{4}OA$, then

$$\angle AOE = \frac{AE}{OA} = \frac{\frac{3}{4}OA}{OA} = \frac{3}{4};$$

and so on.

8. *To find the circular measure of a right angle.*

Before we can do this, it is necessary to know, that in any circle, the

$$\text{circumference} = \text{diameter} \times 3{\cdot}14159\ldots,$$

and that, for convenience' sake, the Greek letter π is usually written instead of $3{\cdot}14159\ldots$. For practical purposes hereafter it will be sufficiently correct to assume that $\pi = 3{\cdot}1416$.

Now let θ be required circular measure. By definition,

$$\theta = \frac{\text{arc subtending right angle in any circle}}{\text{radius of same circle}}$$

$$= \frac{1}{2} \cdot \frac{\text{circumference}}{\text{diameter}} = \frac{1}{2} \times 3{\cdot}1416;$$

$$\text{or } \theta = \frac{\pi}{2}, \quad \text{where } \pi = 3{\cdot}1416.$$

9. Knowing that the angle, which in the sexagesimal measure is 90°, is $\frac{\pi}{2}$ in the circular, it is a mere matter of proportion to connect the two measures.

Thus, if θ be the circular measure of any angle which contains D degrees,

$$90 : D :: \frac{\pi}{2} : \theta.$$

Hence, first, $$\theta = \frac{\pi}{180} D;$$

and secondly, $$D^{\circ} = \frac{180}{\pi} \theta.$$

Writing G for the number of grades in any circle, the student himself should connect G with θ.

10. It is usual to indicate that an angle is supposed to be measured in the sexagesimal measure by using one of the Roman letters, A, B, C, &c., to represent it; and in the circular measure by putting for it one of the Greek letters, α, β, γ, &c.

11. (1.) Find the circular and centesimal measures of 25°.

Call the required circular measure θ; then beginning, as is

preferable in the case of learners, with the statement of the proportion,

$$90 : 25 :: \frac{\pi}{2} : \theta,$$

$$\therefore \quad \theta = \frac{25\pi}{90 \times 2} = \frac{5}{36} \times 3{\cdot}1416 = {\cdot}4363.$$

For the centesimal measure we have the formula

$$G = D + \frac{D}{9}.$$

Therefore the required number of grades

$$= 25^g + \frac{25^g}{9} = 25^g + 2^g{\cdot}77777\ldots$$

$$= 27^g\ 77'\ 77''{\cdot}8 \text{ nearly.}$$

(2.) What are the sexagesimal and centesimal measures of the angle whose circular measure is $\frac{3\pi}{8}$, or 1·1781?

Let D = required number of degrees, &c.,

$$\frac{\pi}{2} : \frac{3\pi}{8} :: 90^\circ : D^\circ,$$

$$\therefore \quad D^\circ = 90^\circ \times \frac{3\pi \times 2}{8 \times \pi} = 67^\circ\ 30'.$$

Let G = required number of grades, &c.,

$$\frac{\pi}{2} : \frac{3\pi}{8} :: 100^g : G^g,$$

$$\therefore \quad G^g = 100^g \times \frac{3\pi \times 2}{8 \times \pi} = 75^g.$$

(3.) Express the unit of circular in the sexagesimal measure.

Otherwise expressed, the degrees &c. are required in the angle whose circular measure is 1.

If x be required number of degrees,

$$\frac{\pi}{2} : 1 :: 90^\circ : x^\circ,$$

$$\therefore \quad x^\circ = \frac{90^\circ \times 2}{\pi} = \frac{180^\circ}{3{\cdot}1416} = 57^\circ{\cdot}29577.$$

Hence the unit of circular measure contains 57° 17′ 44″·8, 206265″ nearly.

Examples.

1. Find in the sexagesimal measure the values of the angles half a right angle, three-fourths of a right angle, seventeen

elevenths of a right angle; and in the centesimal those of the angles, four-thirds of a right angle, and six-sevenths of a right angle.

2. Express in circular measure the same angles.

3. Find the circular measures of the angles 30°, 22° 30′, 156° 45′, and 96° 12′ 12″.

4. Express in circular measure 150^g and $73^g\ 62^{\prime}\ 3^{\prime\prime\prime}{\cdot}7$.

5. Express in the sexagesimal measure the angles whose circular measures are $\frac{5\pi}{6}$, ·5323, ·8447.

6. Find the centesimal measures of the angles which in circular measure are $\frac{\pi}{5}$ and 1·7482.

7. Express in grades &c. the angles 150° and 18° 7′ 14″·16.

8. Express in degrees &c. the angles 125^g and $56^g\ 24^{\prime}\ 25^{\prime\prime}$.

9. Find each angle of a regular triangle, hexagon, decagon, and quindecagon, in degrees and in circular measure.

10. If one-fourth of a right angle be taken as unit, what are the measures of 60° and of 60^g?

11. What number would represent the angle whose circular measure is ·5236, if 45° were assumed as the unit of measure?

12. If G, D, C be respectively the numbers of grades, degrees, and units of circular measure in the same angle, show that

$$\frac{G}{10} = \frac{D}{9} = \frac{2C}{\pi}.$$

CHAPTER II.

TRIGONOMETRICAL FUNCTIONS OF ACUTE ANGLES.

12. CALCULATIONS respecting angles are mostly carried on in Trigonometry by means of certain ratios connected with them, called their *functions*, or the *trigonometrical ratios* of them.

The values of these ratios may be very conveniently expressed when the acute angles of a right-angled triangle are considered; and as such angles only will for the present be treated of, the definitions of the ratios for angles generally will be deferred.

The names of the trigonometrical functions of angles are *sine*, *cosine*, *tangent*, *cotangent*, *secant*, *cosecant*, and *versed-sine;* which are not usually written at full length, however, but as in the following statements concerning them.

13. *The trigonometrical ratios of the acute angles of a right-angled triangle expressed in terms of the sides.*

If in any right-angled triangle we write A for either of its acute angles,

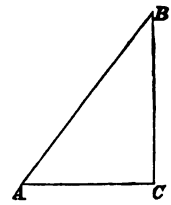
opp. for side opposite to A,
adj. for side adjacent to it, and
hyp. for hypothenuse of triangle; then

$$\sin A = \frac{\text{opp.}}{\text{hyp.}}, \quad \cos A = \frac{\text{adj.}}{\text{hyp.}}, \quad \tan A = \frac{\text{opp.}}{\text{adj.}},$$

$$\operatorname{cosec} A = \frac{\text{hyp.}}{\text{opp.}}, \quad \sec A = \frac{\text{hyp.}}{\text{adj.}}, \quad \cot A = \frac{\text{adj.}}{\text{opp.}},$$

$$\text{versin A} = 1 - \cos \text{A}.$$

Thus, if ABC be a right-angled triangle,



$$\sin A = \frac{BC}{AB}, \quad \sin B = \frac{AC}{AB},$$

$$\cos A = \frac{AC}{AB}, \quad \cos B = \frac{BC}{AB},$$

$$\tan A = \frac{BC}{AC}, \quad \tan B = \frac{AC}{BC},$$

and so for the remaining ratios.

14. *Relations between the Functions of Angles.*—There are certain relations of great importance existing between these ratios; and the five results of the demonstrations which follow should be carefully noted, and committed to memory as formulæ.

In the right-angled triangle ABC,

$$\sin A \times \operatorname{cosec} A = \frac{BC}{AB} \times \frac{AB}{BC} = 1;$$

$$\therefore \sin A = \frac{1}{\operatorname{cosec} A}, \text{ and } \operatorname{cosec} A = \frac{1}{\sin A} \quad \ldots(1).$$

$$\cos A \times \sec A = \frac{AC}{AB} \times \frac{AB}{AC} = 1;$$

$$\therefore \cos A = \frac{1}{\sec A}, \text{ and } \sec A = \frac{1}{\cos A} \quad \ldots\ldots\ldots(2).$$

$$\tan A \times \cot A = \frac{BC}{AC} \times \frac{AC}{BC} = 1;$$

$$\therefore \tan A = \frac{1}{\cot A}, \text{ and } \cot A = \frac{1}{\tan A} \quad \ldots\ldots\ldots(3).$$

$$\tan A = \frac{BC}{AC}, \text{ and } \frac{\sin A}{\cos A} = \frac{BC}{AB} \div \frac{AC}{AB} = \frac{BC}{AC};$$

$$\therefore \tan A = \frac{\sin A}{\cos A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

$$\cot A = \frac{AC}{BC}, \text{ and } \frac{\cos A}{\sin A} = \frac{AC}{AB} \div \frac{BC}{AB} = \frac{AC}{BC};$$

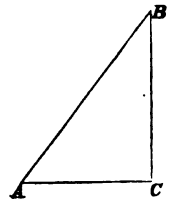
$$\therefore \cot A = \frac{\cos A}{\sin A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Or thus, in this last case, applying (3),

$$\cot A = \frac{1}{\tan A} = \frac{\cos A}{\sin A}.$$

15. The squares of $\sin A$, $\cos A$, $\tan A$, &c., or $(\sin A)^2$, $(\cos A)^2$, $(\tan A)^2$, &c., are generally written for the sake of shortness thus: $\sin^2 A$, $\cos^2 A$, $\tan^2 A$, &c.

The following results, in which the squares of the ratios are involved, are of equal importance with those of the last article, and must be added to them as formulæ to be well learnt.



Since by Euc. I. 47, in the right-angled triangle ABC,

$$BC^2 + AC^2 = AB^2;$$

dividing by AB^2,

$$\left(\frac{BC}{AB}\right)^2 + \left(\frac{AC}{AB}\right)^2 = 1,$$

or, $$\sin^2 A + \cos^2 A = 1.$$

$$\left.\begin{array}{l} \therefore\ \sin^2 A = 1 - \cos^2 A, \text{ and } \sin A = \sqrt{1-\cos^2 A} \\ \text{also } \cos^2 A = 1 - \sin^2 A, \text{ and } \cos A = \sqrt{1-\sin^2 A} \end{array}\right\} \quad (6).$$

Again, since in ABC,

$$AB^2 = BC^2 + AC^2;$$

dividing by AC^2,

$$\left(\frac{AB}{AC}\right)^2 = \left(\frac{BC}{AC}\right)^2 + 1;$$

$$\left.\begin{array}{l} \therefore\ \sec^2 A = \tan^2 A + 1, \text{ and } \sec A = \sqrt{\tan^2 A + 1} \\ \therefore\ \tan^2 A = \sec^2 A - 1, \text{ and } \tan A = \sqrt{\sec^2 A - 1} \end{array}\right\} \quad (7).$$

And since in ABC,

$$AB^2 = AC^2 + BC^2,$$

dividing by BC^2,

$$\left(\frac{AB}{BC}\right)^2 = \left(\frac{AC}{BC}\right)^2 + 1;$$

$$\left.\begin{array}{l} \therefore\ \operatorname{cosec}^2 A = \cot^2 A + 1, \text{ and } \operatorname{cosec} A = \sqrt{\cot^2 A + 1} \\ \text{also } \cot^2 A = \operatorname{cosec}^2 A - 1, \text{ and } \cot A = \sqrt{\operatorname{cosec}^2 A - 1} \end{array}\right\} \quad (8).$$

16. The following are examples of exercises in which the results of the present chapter are applied.

(1.) Prove that $\sin A = \cos A \tan A$.

$$\cos A \tan A = \cos A \frac{\sin A}{\cos A} = \sin A.$$

(2.) Express $\cos A$ in terms of $\cot A$.

$$\cos^2 A = 1 - \sin^2 A = 1 - \frac{1}{\operatorname{cosec}^2 A}$$

$$= \frac{\operatorname{cosec}^2 A - 1}{\operatorname{cosec}^2 A} = \frac{\cot^2 A}{\cot^2 A + 1};$$

$$\therefore \quad \cos A = \frac{\cot A}{\sqrt{\cot^2 A + 1}}.$$

This expression for $\cos A$ is said to be in terms of $\cot A$, because it contains no trigonometrical ratio but $\cot A$. It may be useful to find it by other methods; thus,

$$\cos A = \frac{\cos A}{\sin A} \sin A = \cot A \sin A.$$

$$= \frac{\cot A}{\operatorname{cosec} A} = \frac{\cot A}{\sqrt{\cot^2 A - 1}}.$$

Also,

$$\because \quad \frac{\cos^2 A}{\sin^2 A} = \cot^2 A,$$

$$\therefore \quad \frac{\cos^2 A}{1 - \cos^2 A} = \cot^2 A;$$

therefore, by theory of proportion ("componendo"),

$$\cos^2 A = \frac{\cot^2 A}{\cot^2 A + 1};$$

$$\cos A = \frac{\cot A}{\sqrt{\cot^2 A + 1}}.$$

(3.) If $\sin A = \frac{1}{2}$, find the numerical value of $\tan A$.

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{1 - \sin^2 A}} = \frac{\frac{1}{2}}{\sqrt{\left(1 - \frac{1}{4}\right)}} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}};$$

$$\therefore \quad \tan A = \frac{1}{\sqrt{3}}.$$

(4.) Prove the formula

$$\sin^6 \theta + \cos^6 \theta + 3 \sin^2 \theta \cos^2 \theta = 1.$$

Resolving a portion of the expression into factors, and applying the formula $\sin^2 A + \cos^2 A = 1$,

$$\sin^6\theta + \cos^6\theta + 3\sin^2\theta\cos^2\theta.$$
$$= (\sin^2\theta + \cos^2\theta)(\sin^4\theta - \sin^2\theta\cos^2\theta + \cos^4\theta) + 3\sin^2\theta\cos^2\theta$$
$$= \sin^4\theta - \sin^2\theta\cos^2\theta + \cos^4\theta + 3\sin^2\theta\cos^2\theta$$
$$= (\sin^2\theta + \cos^2\theta)^2 = 1.$$

EXAMPLES.

Prove the truth of the formulæ in 1—5.

1. $\sin A = \cos A \tan A$; and $\cos A = \sin A \cot A$.
2. $\tan^2 A \cot A = \tan A$; and $\tan A = \dfrac{\sec A}{\operatorname{cosec} A}$.
3. $\sec A = \sqrt{\sec^2 A \sin^2 A + 1}$.
4. $\cot A = \operatorname{cosec} A \sqrt{1 - \sin^2 A}$.
5. $\dfrac{\cos A}{\sin A \cot^2 A} = \tan A$.
6. Express $\sin A$ in terms of $\tan A$, and $\cos A$ in terms of $\operatorname{cosec} A$.
7. Express $\tan A$ in terms of $\cos A$, and $\cot A$ in terms of $\sec A$.
8. Express $\sec A$ in terms of $\sin A$, and $\operatorname{cosec} A$ in terms of versin A.
9. Given $\tan A = \frac{3}{4}$, find all the other trigonometrical ratios of A.
10. Given $\sin A = \cdot 5$, find $\tan A$, $\cos A$, versin A.
11. Given $\sec A = 2\sqrt{2}$, find $\sin A$, $\cot A$.

Prove the formulæ in 12—31.

12. $1 + 2\sin\theta\cos\theta = (\sin\theta + \cos\theta)^2$.
13. $\tan\theta + \cot\theta = \sec\theta\operatorname{cosec}\theta$.
14. $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta\operatorname{cosec}^2\theta$.
15. $\cot^2\theta - \cos^2\theta = \cot^2\theta\cos^2\theta$.
16. $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$.
17. $(\sec\theta\cot\theta)^2 + (\operatorname{cosec}\theta\tan\theta)^2 = (\sec\theta\operatorname{cosec}\theta)^2$.
18. $(\cos\theta\tan\theta)^2 + (\sin\theta\cot\theta)^2 = 1$.
19. $\sin A(1 + \tan A) + \cos A(1 + \cot A) = \operatorname{cosec} A + \sec A$.
20. $\dfrac{(\operatorname{cosec} A + \sec A)^2}{\operatorname{cosec}^2 A + \sec^2 A)} = 1 + 2\sin A\cos A$.
21. $\dfrac{\sec A\cot A - \operatorname{cosec} A\tan A}{\cos A - \sin A} = \sec A\operatorname{cosec} A$.

22. $\sec^2 A \tan^2 A (\sec^2 B - 1)(\tan^2 B + 1) =$
$\sec^2 B \tan^2 B (\sec^2 A - 1)(\tan^2 A + 1)$.

23. $\dfrac{\tan A + \tan B}{\cot A + \cot B} = \tan A \tan B$.

24. $\sin^2 A \cos^2 B - \cos^2 A \sin^2 B = \sin^2 A - \sin^2 B$.

25. $\cos^2 A \cos^2 B - \sin^2 A \sin^2 B = \cos^2 A - \sin^2 B$.

26. $2(\sin^6 \theta + \cos^6 \theta) - 3(\sin^4 \theta + \cos^4 \theta) + 1 = 0$.

27. $\sin^2 \theta \tan \theta + \cos^2 \theta \cot \theta + 2 \sin \theta \cos \theta = \tan \theta + \cot \theta$.

28. $\tan^2 A \tan^2 B - 1 = \dfrac{\sin^2 A - \cos^2 B}{\cos^2 A \cos^2 B}$.

29. $\dfrac{1 - \tan^2 A \tan^2 B}{\tan^2 A \tan^2 B} = \dfrac{\cos^2 A - \sin^2 B}{\sin^2 A \sin^2 B}$.

30. $\sec^4 \theta + \tan^4 \theta = 1 + 2 \sec^2 \theta \tan^2 \theta$.

31. $\cot \theta (1 - \cos \theta) + \tan \theta = \operatorname{cosec} \theta (\sec \theta - 1) + \sin \theta$.

32. If $\tan \theta + \sin \theta = m$, and $\tan \theta - \sin \theta = n$; then
$$m^2 - n^2 = 4\sqrt{mn}.$$

33. Show that if C be the chord subtending an angle 2α at the centre of a circle of radius r, then $C = 2r \sin \alpha$.

CHAPTER III.

COMPLEMENTS.—NUMERICAL VALUES OF CERTAIN RATIOS.

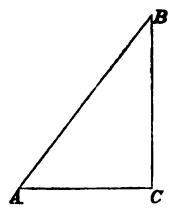
17. *Def.*—If the sum of two angles be a right angle, each of them is called the *complement* of the other.

The same thing is expressed by saying that A and $90°-A$ are complementary to each other; or that α and β are complementary when $\alpha+\beta=\frac{\pi}{2}$.

It follows that, in any right-angled triangle ABC, in which C is the right angle, A and B are complements; for by Euc. I. 32,

$$A+B=90°.$$

18. We shall now obtain additional formulæ worthy of careful attention; numbering some which should be committed to memory. From these the others are easily deduced whenever required.



Let A and B be the acute angles of any right-angled triangle ABC; then

$$\sin A=\frac{BC}{AB}, \text{ and } \cos B=\frac{BC}{AB};$$

$$\left.\begin{array}{ll} \therefore \sin A=\cos B, \text{ or } & \sin A=\cos(90°-A). \\ \text{Similarly} & \cos A=\sin(90°-A). \end{array}\right\} \ldots\ldots(1).$$

The same mode of proof will also show that

$\tan A=\cot(90°-A)$, and $\cot A=\tan(90°-A)\ldots(2)$,

$\sec A=\operatorname{cosec}(90°-A)$, and $\operatorname{cosec} A=\sec(90°-A)\ldots(3)$.

Or we might have proceeded in these last cases after the following manner:—

$$\tan A = \frac{\sin A}{\cos A} = \frac{\cos (90^\circ - A)}{\sin (90^\circ - A)} = \cot (90^\circ - A);$$

and so for the rest.

The designations *co*sine, *co*tangent, *co*secant, have their origin in the facts just proved.

19. *To find the numerical values of the trigonometrical ratios of the angle* 45°.

Since when A is any angle, $\sin^2 A + \cos^2 A = 1$,

$$\therefore \sin^2 45^\circ + \cos^2 45^\circ = 1;$$

but $$\cos 45^\circ = \sin (90^\circ - 45^\circ) = \sin 45^\circ;$$

$$\therefore 2 \sin^2 45^\circ = 1, \text{ and } \sin^2 45^\circ = \frac{1}{2};$$

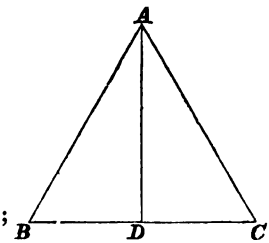
$$\left.\begin{aligned} \therefore \sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}} \\ \therefore \tan 45^\circ = \cot 45^\circ = 1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (4);$$

for $$\tan 45^\circ = \frac{\sin 45^\circ}{\cos 45^\circ} = \frac{\sin 45^\circ}{\sin 45^\circ}.$$

In a similar way we show that

$$\sec 45^\circ = \operatorname{cosec} 45^\circ = \sqrt{2}.$$

20. *To find the values of the trigonometrical ratios of* 30° *and* 60°.



Let the triangle ABC be equilateral, and let AB be drawn bisecting the angle BAC; then by Euc. I., Props. 4, 32, it is easily shown that

$$AB = BC = 2BD,$$

and $$\angle BAD = \frac{1}{2} \angle BAC = 30^\circ;$$

$$\therefore \sin 30^\circ = \frac{BD}{AB} = \frac{BD}{2BD} = \frac{1}{2}.$$

Hence $\cos 30^\circ = \sqrt{1-\sin^2 30^\circ} = \frac{\sqrt{3}}{2}$;

and $\tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ} = \frac{1}{\sqrt{3}}$.

The values of $\cot 30^\circ$, $\sec 30^\circ$, $\operatorname{cosec} 30^\circ$ are obvious.

Since $\sin A = \cos (90^\circ - A)$,

we have from the above

$$\left.\begin{aligned} \sin 30^\circ &= \cos 60^\circ = \frac{1}{2} \\ \sin 60^\circ &= \cos 30^\circ = \frac{\sqrt{3}}{2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

21. The following are instances of Exercises on the foregoing formulæ:—

(1.) Prove that $\tan a = \cos\left(\frac{\pi}{2}-a\right) \operatorname{cosec}\left(\frac{\pi}{2}-a\right)$.

$$\cos\left(\frac{\pi}{2}-a\right) \operatorname{cosec}\left(\frac{\pi}{2}-a\right) = \sin a \sec a = \frac{\sin a}{\cos a} = \tan a.$$

(2.) Find A, when $\sin A = \cos 2A$.

$$\because \cos 2A = \sin (90^\circ - 2A),$$
$$\therefore \sin A = \sin (90^\circ - 2A);$$
$$\therefore A = 90^\circ - 2A, \text{ and } 3A = 90^\circ;$$
$$\therefore A = 30^\circ.$$

(3.) Show that $\frac{\cos 45^\circ + \cos 30^\circ}{\cos 45^\circ - \cos 60^\circ} = \frac{\sec 45^\circ + \cot 30^\circ}{\sec 45^\circ - \cot 45^\circ}$.

$$\frac{\cos 45^\circ + \cos 30^\circ}{\cos 45^\circ - \cos 60^\circ} = \frac{\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}}{\frac{1}{\sqrt{2}} - \frac{1}{2}}$$

$$= \frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}-1} = \frac{\sec 45^\circ + \cot 30^\circ}{\sec 45^\circ - \cot 45^\circ}.$$

(4.) Solve the equation $\sec \theta = 2 \tan \theta$.

Obs. If there be more functions than one involved in the equation, transform until there is one only; that function is the unknown quantity of the equation. The remaining part of the process is obvious from the Examples.

$$\sec \theta = 2 \tan \theta,$$
$$\therefore \frac{1}{\cos \theta} = \frac{2 \sin \theta}{\cos \theta};$$

multiplying by $\cos\theta$ and transposing,

$$2\sin\theta = 1,$$

$$\therefore \sin\theta = \frac{1}{2} = \sin 30^\circ;$$

$$\therefore \theta = 30^\circ.$$

(5.) If $\tan\theta + 3\cot\theta = 4$, find θ.

$$\tan\theta + 3\cot\theta = 4,$$

or

$$\tan\theta + \frac{3}{\tan\theta} = 4;$$

clearing of fractions and transposing,

$$\tan^2\theta - 4\tan\theta = -3;$$

completing square,

$$\tan^2\theta - 4\tan\theta + 4 = -3 + 4 = 1;$$

extracting root and transposing,

$$\tan\theta = \pm 1 + 2 = 3 \text{ or } 1;$$

and since

$$1 = \tan 45^\circ,$$

one value of θ is 45°.

Obs. A subsequent chapter would enable us to state the other value of θ, but chapters I.—III. do not. A similar remark will apply to the results of most of the equations for exercise which immediately follow.

Examples.

1. What are the complements of the angles 12°, $25^\circ\ 4'$, $56^\circ\ 50'\ 44''$, $88^\circ\ 58'\ 58''{\cdot}89$?

2. Show that $45^\circ \pm A$ is the complement of $45^\circ \mp A$.

Prove the formulæ in 3—6.

3. $\sin A \operatorname{cosec}(90^\circ - A) = \cot(90^\circ - A)$.

4. $\tan. A = \dfrac{\sec A}{\sec(90^\circ - A)}$.

5. $\sec a \operatorname{versin} a = \operatorname{cosec}\left(\dfrac{\pi}{2} - a\right) - 1$.

6. $\sin\left(\dfrac{\pi}{2} - \theta\right)\cos\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{1}{\tan\left(\dfrac{\pi}{2} - \theta\right) + \cot\left(\dfrac{\pi}{2} - \theta\right)}$.

7. If $\tan A = \cot A$, what is the numerical value of A?

8. When $\sin 5A = \cos A$, find A.

9. When $\sec 3A = \operatorname{cosec} 2A$, find A.

10. Write out the numerical values of the secants, cosecants, cotangents, and versed sines of the angles $30°, 45°, 60°$; and prove *geometrically* that

$$\sin 45° = \cos 45° = \frac{1}{\sqrt{2}}.$$

11. Show that $\sin^2 30° : \sin^2 45° : \sin^2 60° :: 1 : 2 : 3$. Also prove *geometrically* that $\cot 30° = \sqrt{3}$.

Verify the equations in 12—17.

12. $\dfrac{1-\tan^2 30°}{1+\tan^2 30°} = \cos 60°.$

13. $\cot 30° = \dfrac{2 \tan 30°}{1-\tan^2 30°}.$

14. $4 \cos^3 60° - 3 \cos 60° = -1.$

15. $\dfrac{\sin 45° - \sin 30°}{\sin 45° + \sin 30°} = \dfrac{\sec 45° - \tan 45°}{\sec 45° + \tan 45°}.$

16. $\dfrac{\sin 60° + \sin 30°}{\sin 60° - \sin 30°} = \dfrac{\tan 60° + \tan 45°}{\tan 60° - \tan 45°}.$

17. $\dfrac{3 \tan 30° - \tan^3 30°}{1 - 3 \tan^2 30°} = \infty.$

Obtain values for θ satisfying the equations in 18—27.

18. $2 \sin \theta = \operatorname{cosec} \theta.$

19. $\tan \theta + \cot \theta = 2 \sec \theta.$

20. $2 \tan^2 \theta = \sec^2 \theta.$

21. $2 \sin^2 \theta + 5 \cos \theta = 4.$

22. $\tan \theta + \sec^2 \theta = 3.$

23. $\cos^2 \theta - \sin^2 \theta = \dfrac{1}{2}.$

24. $3 \cos \theta - 2 \cos^2 \theta = 1.$

25. $3 \cos \theta + 4 \sin^2 \theta = 6.$

26. $4 \sin^2 \theta + 3 \operatorname{cosec}^2 \theta = 7.$

27. $5 \operatorname{cosec} \theta - 2 \operatorname{cosec}^2 \theta = 2.$

28. If $4 \tan A = \cot A$, find $\tan A$.

29. If $25 \sin^2 A = 16$, find $\cos A$.

30. $3 \sin^2 \alpha + 2 \sin \alpha = 1$; required $\sin \alpha$.

31. $\tan A + \cot A = 4$; required $\tan A$.

32. $\tan^2 \theta = 3 \tan \theta + 4\sqrt{\tan \theta} + 3$; required $\tan \theta$.

33. Solve the simultaneous equations,

$$\sin A = \sqrt{2} \sin B, \ \tan A = \sqrt{3} \tan B.$$

34. Solve $\tan (A+B) = \sqrt{3}$, $\cot (A-B) = \sqrt{3}$.

35. Find α and β, when

$$\sin \alpha + \sin \beta = \sqrt{2}, \text{ and } \sin \alpha \sin \beta = \frac{1}{2}.$$

36. Find A and B, when

$$\cos A + \cos B = \frac{1+\sqrt{3}}{2}, \quad \cos A \cos B = \frac{\sqrt{3}}{4}.$$

37. If $m \sin A = n \cos A$, find $\sin A$.

38. Given $\tan \alpha + \sin \alpha = m$, $\tan \alpha - \sin \alpha = n$; find $\sec \alpha$.

39. Given $\cos \theta = \dfrac{\cos A}{\sin C}$, and $\cos (90^\circ - \theta) = \dfrac{\cos B}{\sin C}$; show that

$$\sin^2 A + \sin^2 B + \sin^2 C = 2.$$

40. If $\tan \theta + \cot \theta = 2$, show that $\sin \theta + \cos \theta = \sqrt{2}$.

41. If $\cos \phi = n \sin A$, and $\cot \phi = \dfrac{\sin A}{\tan B}$, then

$$\cos \beta = \frac{n}{\sqrt{1 + n^2 \cos^2 A}}.$$

42. Given $\tan \theta = \dfrac{m}{n} \cos A$, and $\sin \theta = \dfrac{\cos A}{\sin B}$, then

$$\sin B = \pm \frac{\sqrt{n^2 + m^2 \cos^2 A}}{m}.$$

CHAPTER IV.

SOLUTION OF RIGHT-ANGLED TRIANGLES.

22. The angles and sides of any triangle, six in number, are spoken of as the *parts* of that triangle. It is usual to denote the three angles by the capitals A, B, C, and the three sides respectively opposite to A, B, C by the small letters a, b, c. When the triangle contains a right angle, we shall most commonly call it C, and hence the hypothenuse c.

In the case of the right-angled triangle two of the parts being given besides the right angle, the other three, or one or two as may be required, can always be found. The solution of problems in which the parts of a triangle are required is what is meant by the "solution of triangles."

The mode adopted in Art. 13, of expressing the trigonometrical ratios for acute angles will now be found useful, and should be effectually impressed upon the memory. In addition, Euc. I. 47. will be required.

EXERCISE II. — (1.) ABC being a right-angled triangle, write out the functions of the acute angles A and B in terms of a, b, c. Ans. $\sin A = \frac{a}{c}$, $\cos A = \frac{b}{c}$, &c.; $\sin B = \frac{b}{c}$, &c.

(2.) Prove the truth of the following equations:—

(i.) $a = c \cos B$.

(ii.) $b^2 = ac \cos A \tan B$.

(iii.) $c^3 = ab^2 \sec A \sec B \operatorname{cosec} B$.

(iv.) $\dfrac{\sin A + \cos A}{\sin B} = \dfrac{a+b}{b}$.

(v.) $\dfrac{\sin A + \sin B + 1}{\sin A} = \dfrac{a+b+c}{a}$.

(vi.) $\tan^2 A - \cot^2 A = \dfrac{a^4 - b^4}{a^2 b^2}$.

23. Of the solution of problems, of whatever kind, much is most easily learnt by the inspection of examples worked out, in the first place; and in the second, by observation and experience. Accordingly we shall give but few hints in this part of our subject otherwise than by actual solution of triangles. Those few, however, will be worthy of attention.

24. The application of Euc. I. 47, is at all times so obvious as to require no separate illustration. With the aid of it, any two sides of a right-angled triangle being given, we find the remaining side; for

$$c^2 = a^2 + b^2, \text{ or } c = \sqrt{a^2+b^2};$$
$$\therefore\ a^2 = c^2 - b^2, \text{ or } a = \sqrt{c^2-b^2};$$
$$\text{and } b^2 = c^2 - a^2, \text{ or } b = \sqrt{c^2-a^2}.$$

The same may be said of the formula $A + B = 90°$, by which the value of one acute angle determines that of the other.

The operations which require exemplification in the cases of the solution of triangles will be treated of under two heads.

25. *To find a side, when an angle and a side are given.*

Forming a fraction having the required side as numerator and the given side as denominator, we find that we have one of the trigonometrical ratios of the given angle. Equate the fraction and the ratio; then a simple equation results, from which we may determine the unknown side.

For instance, it is required to find a, when the values of B and c are known.

The fraction formed as described is $\dfrac{a}{c}$, and this, by Art. 13, is $\cos B$. Equating, and clearing the fraction,

$$\frac{a}{c} = \cos B, \quad \therefore\ a = c \cos B,$$

from which a is known.

Again, to find c, when A and a are known. Proceeding as before,

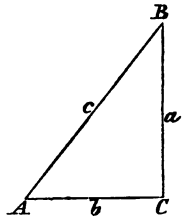
$$\frac{c}{a} = \operatorname{cosec} A, \quad \therefore \; c = a \operatorname{cosec} A.$$

A result like the last, involving sec A, cosec A, or versin A, might be modified, if useful or necessary, thus:

$$c = a \frac{1}{\sin A} = \frac{a}{\sin A}.$$

26. The following are numerical examples:—

(1.) Given in ABC, $A = 60°$, $c = 40$ ft.; required a.



$$\frac{a}{c} = \sin A, \quad \therefore \; a = c \sin A;$$

$$\therefore \; a = 40 \text{ ft.} \times \frac{\sqrt{3}}{2} = 20\sqrt{3} \text{ ft.}$$

$$= 34{\cdot}64 \text{ ft.}$$

Obs.—The value of $\sqrt{3}$ is here taken to be 1·732, which is nearly its real value.

(2.) Given $B = 30°$, $a = 15$ feet; required to solve the triangle *completely*; that is, to find all the remaining parts, A, b, c.

(i.) $A + B = 90°$, $\therefore A = 90° - 30° = 60°$.

(ii.) $\frac{b}{a} = \tan B$, $\therefore b = a \tan B = a \tan 30°$.

$$\therefore \; b = 15 \text{ ft.} \times \frac{1}{\sqrt{3}} = 5\sqrt{3} \text{ ft.}$$

(iii.) $c^2 = a^2 + b^2 = 225$ sq. ft. $+ 75$ sq. ft. $= 300$ sq. ft.

$$\therefore \; c = 10\sqrt{3} \text{ ft.}$$

27. *To find an angle when two sides are given.*

Dividing one given side by the other, the fraction thus formed will be some function of the angle required. Equate the fraction and the function; and the equation which results will determine the angle. The manner in which it does so, however, can be *clearly* shown only by the working of Examples.

If a, c be given, and A required, the equation will be

$$\sin A = \frac{a}{c}, \text{ or } \operatorname{cosec} A = \frac{c}{a};$$

either of which determines A.

Secants and cosecants will be avoided in these equations, if it be remembered, when c is one of the given sides, to make it the *denominator* of the fraction.

Let a, b be given, and B required. Here we have

$$\frac{a}{b} = \cot B, \text{ or } \frac{b}{a} = \tan B.$$

(3.) Given $a = 6$, $b = 6$, required to solve the triangle.

The following three methods are given, in order both to exemplify fully the present case, and to illustrate the variety of methods of setting to work usually open to the student.

First Method :—

(i.) $c^2 = a^2 + b^2 = 72, \quad \therefore c = 6\sqrt{2}.$

(ii.) $\sin A = \frac{a}{c} = \frac{6}{6\sqrt{2}} = \frac{1}{\sqrt{2}} = \sin 45°;$

$\therefore A = 45°.$

(iii.) $B = 90° - 45° = 45°.$

Second Method :—

(i.) $\tan A = \frac{a}{b} = \frac{6}{6} = 1 = \tan 45°;$

$\therefore A = 45°, \quad \therefore$ also $B = 45°.$

(ii.) $\frac{c}{a} = \sec B = \frac{1}{\cos B};$

$\therefore c = \frac{a}{\cos B} = \frac{6}{\cos 45°} = 6\sqrt{2}.$

Third Method :—

(i). $a = b$; hence, by Euc. I. 5, $A = B$,

and $A + B = 90°$;

$\therefore A = B = 45°.$

(ii.) $c^2 = a^2 + b^2$, as before.

28. In the examples so far, the three angles only whose numerical values have been determined have been employed. Such as the following differ in this respect :—

(4.) Given $a = 24{\cdot}6$, $c = 41$; required A and B; when $\sin 36° 52' = {\cdot}6$.

$$\frac{a}{c} = \sin A, \quad \therefore \sin A = \frac{24{\cdot}6}{41} = {\cdot}6 = \sin 36° 52';$$

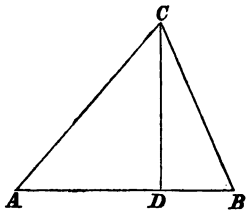
$$\therefore A = 36° 52';$$

and $$\therefore B = 90° - 36° 52' = 53° 8'.$$

29. Examples will now be given of exercises which involve the construction of more right-angled triangles than one.

(1.) ABC is a triangle, and CD is perpendicular to AB. Find the length of the side AB, given that

$$CD = 10,\ \angle A = 45°,\ \angle B \doteq 60°.$$



In the right-angled triangle ACD,

$$\angle A = 45°,\quad \therefore\ \angle ACD = 45°;$$

$$\therefore\ BD = DC = 10.$$

In the right-angled triangle BCD,

$$BD = CD \cot B = \frac{10}{\sqrt{3}} = \frac{10\sqrt{3}}{3} = \frac{17{\cdot}32}{3};$$

$$\therefore\ BD = 5{\cdot}77\dot{3}.$$

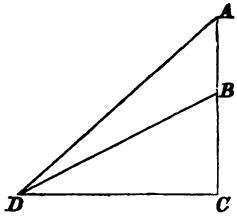
And $$AB = AD + BD = 10 + 5{\cdot}77\dot{3};$$

$$\therefore\ AB = 15{\cdot}77\dot{3}.$$

30. It is sometimes expedient in such problems to solve by means of simultaneous equations; when, of course, as many equations are required as there are unknown quantities involved in the operations.

(2.) CBA is a straight line at right angles to CD; required BC, given that

$$AB = a,\ \angle ADC = \alpha,\ \angle BDC = \beta.$$



Let $x = BC$; CD will appear in our operations, although it is neither known nor required; therefore let $y = CD$.

Then we have the two equations

$$\frac{y}{x+a} = \cot \alpha, \quad \text{or} \quad y = (x+a) \cot \alpha,$$

and $$\frac{y}{x} = \cot \beta, \quad \text{or} \quad y = x \cot \beta;$$

$$\therefore\ x \cot \beta = (x+a) \cot \alpha;$$

transposing, $$x(\cot \beta - \cot \alpha) = a \cot \alpha,$$

$$\therefore\ x = \frac{a \cot \alpha}{\cot \beta - \cot \alpha};$$

which is the required expression for BC.

Otherwise. The result might also have been obtained as follows:—

$$\frac{x}{y} = \tan \beta, \quad \text{or} \quad x = y \tan \beta \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and $$\frac{x+a}{y} = \tan \alpha, \quad \text{or} \quad x + a = y \tan \alpha \ldots\ldots\ldots\ldots\ldots (2).$$

Dividing (1) by (2),

$$\frac{x}{x+a} = \frac{\tan\beta}{\tan\alpha};$$

then by theory of proportion,

$$\frac{x}{a} = \frac{\tan\beta}{\tan\alpha - \tan\beta},$$

$$\therefore\ x = \frac{a\tan\beta}{\tan\alpha - \tan\beta}.$$

The student may show for himself that this expression for BC differs from the one first obtained in form only.

EXAMPLES.

(In Examples 1—14, ABC is a triangle, right-angled at C.)

1. Given $A = 30°$, $c = 5$; find a.
2. Given $B = 60°$, $b = 17{\cdot}32$; find a.
3. Given $A = 45°$, $b = 20$; find c.
4. Given $a = 3$, $c = 2\sqrt{3}$; find B.
5. Given $a = 6$, $b = 3\sqrt{3}$; find A.
6. Given $b = 5\sqrt{2}$, $c = 10$; find B.

In 7—10 solve ABC, assuming that

$$\sqrt{2} = 1{\cdot}414,\ \sqrt{3} = 1{\cdot}732;$$

and having given,

7. $A = 30°$, $c = 50$.
8. $B = 30°$, $a = 1732$.
9. $a = 25$, $b = 43{\cdot}3$.
10. $b = 36$, $c = 50{\cdot}904$.
11. When $A = 8°\ 20'\ 14''{\cdot}2$, $c = 100$; find a.
    Given $\sin 8°\ 21'\ 14''{\cdot}2 = {\cdot}145$.
12. When $B = 75°\ 38'\ 25''$, $b = 3{\cdot}2$; find a.
    Given $\cot 75°\ 38'\ 25'' = {\cdot}256$.
13. When $b = 13$, $c = 25$; find A.
    Given $\cos 51°\ 41' = {\cdot}62$.
14. When $a = 8333{\cdot}\dot{3}$, $b = 5529{\cdot}\dot{6}$; find B.
    Given $\tan 33°\ 34' = {\cdot}66356$.
15. ABC is a triangle, and CD is perpendicular to AB. Find CA, CB, and AB, having given

$$CD = 433,\ \angle ACD = 60°,\ \angle BCD = 30°.$$

16. PH is perpendicular to the straight line PQR; find PH and PQ, given that

$$AR = 73{\cdot}2,\ \angle PQH = 45^\circ,\ \angle PRH = 30^\circ.$$

17. CBA is a straight line perpendicular to CD; required AC, given that

$$BC = b,\ \angle ADC = \alpha,\ \angle BDC = \beta.$$

18. CBA is as before; required CD, given that

$$AB = a,\ \angle ADC = \alpha,\ \angle BDC = \beta.$$

19. CBA is as before; find angle ADB, having given

$$AC = 86{\cdot}6,\ BD = 70{\cdot}7,\ \angle BDC = 45^\circ.$$

20. ABC is an equilateral triangle, and AB is produced to D, making $BD = BC$. Find CD, given that $AB = a$.

CHAPTER V.

PROBLEMS ON SOLUTION OF RIGHT-ANGLED TRIANGLES.

31. We may now turn the preceding chapters to practical account in solving easy exercises concerning heights and distances.

The only new process is the construction of diagrams to illustrate each problem. If these be made as clear and as accurate as is consistent with expedition, the care bestowed on them will be amply compensated when they become difficult in any degree.

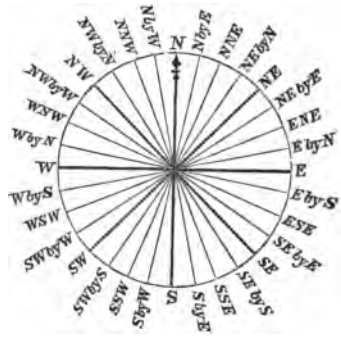
32. The student may require explanations of certain terms which will now come into frequent use.

The *angular altitude*, or *angle of elevation*, of an object *above* the eye of an observer, is the angle contained by two straight lines drawn from the eye, one of which is horizontal, and the other meets the object. The words, *altitude* and *elevation*, simply, are most generally used in Trigonometry to denote this angle.

The *angle of depression*, or *depression* simply, of an object *below* the eye of an observer, is the angle contained by two straight lines drawn from the eye, one of which is horizontal, and the other meets the object.

33. *Mariner's Compass.*—It is useful to know accurately what are called the Points of the Compass. They will be learnt best from the inspection of a "compass card;" a representation of which is here given.

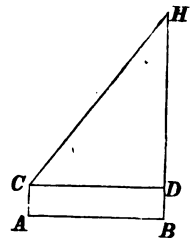
It may also be useful to observe that the meaning of such phrases as N. 30° E., or 30° E. of or from N., is that direction which is between N. and E. and inclined to N. at an angle of 30°. Similarly, on the card, S. W. is S. 45° W.; and E. by S. is $11\frac{1}{4}°$ S. of E.



34. After those of the last chapter, but few examples worked out will be required here.

(1.) At a distance of 36 yards a tower subtends an angle of 60°; and the observer's eye is 6 feet above the ground. What is the height of the tower?

Making BH the tower, and AC the observer, we have given, $AB = 36$ yards, $AC = 6$ feet $= 2$ yards, and $\angle DCH = 60°$; required BH.



Draw CD parallel to AB.

In triangle DCH, because $\frac{HD}{CD} = \tan DCH$,

$\therefore\ HD = CD \tan 60° = 36\sqrt{3}$ yds.,

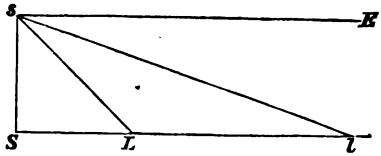
$\therefore\ BH = BD + DH = 2$ yds. $+ 36\sqrt{3}$ yds.;

$\therefore$ required height $= 64{\cdot}352$ yds.

(2.) A ship sailing due N. observes at noon two lighthouses in a line exactly E. At 20 minutes to 1 o'clock they are respectively S.E. and E. by S. At what rate does the ship sail, the lighthouses being 24 miles apart?

Given $\tan 11° 15' = {\cdot}2$ nearly.

Let S, s be positions of ship at noon and at 20 minutes to 1 respectively; and L, l the lighthouses.



Given in question,

$\angle SsL = 45° = SLs$, $\angle Sls = lsE = 11° 15'$, $Ll = 24$ miles;

required Ss ($= x$ miles suppose).

Evidently $LS = Ss = x$, and $lS = 24 + x$;

therefore, $$\frac{x}{24+x} = \tan 11^\circ 15' = \cdot 2 = \frac{1}{5}.$$

$$\therefore\ 5x = 24 + x;$$

$$\therefore\ x = 6.$$

Hence required rate of sailing = 6 miles per 40 minutes
= 9 miles per hour.

(3.) In ABC, having C a right angle, A and $a+b-c$ are given; solve the triangle.

Let $a+b-c = m$;

$$\because\ a = c \sin A, \text{ and } b = c \cos A;$$

$$\therefore\ m = c \sin A + c \cos A - c = c(\sin A + \cos A - 1);$$

and $$\therefore\ c = \frac{m}{\sin A + \cos A - 1},$$

or $$c = \frac{a+b-c}{\sin A + \cos A - 1};$$

$$\therefore\ a = \frac{(a+b-c)\sin A}{\sin A + \cos A - 1},$$

and $$b = \frac{(a+b-c)\cos A}{\sin A + \cos A - 1}.$$

EXAMPLES.

1. How far must a person recede from a pillar 45 feet high, in order that it may subtend at his foot an angle of 60°?

2. The altitude of a tower, observed at the end of a horizontal base of 100 yards measured from its foot, is 30°; find its height.

3. A lighthouse was observed by a ship at sea to bear S.E.; after the ship had sailed N.E. for 12 miles, the lighthouse was observed to bear 15° E. of S. Find the distance of the lighthouse from each position of the ship.

4. The angle of elevation of the top of a steeple is 60° from a point on the ground. That of the top of the tower on which the steeple rests is 45° from the same point. What proportion does the height of the steeple bear to that of the tower?

5. The elevation of the top of a steeple is 45° from a point in the same level as its base, and is 30° from a point 30 feet directly above the former point; find the height and distance of the steeple.

6. A pole is fixed on the top of a mound, and the angles of elevation of the bottom and top of the pole are 30° and 60° re-

spectively; prove that the height of the pole is twice the height of the mound.

7. Two straight roads, which cross one another, meet a canal at angles of 60° and 30° respectively. If it be 3 miles by the longer of the two roads, from the crossing to the canal; how far is it by the shorter? If there be a footpath which goes the shortest way to the canal, what is the distance by it?

8. A ladder 30 feet long reaches a window 24 feet high on one side of a street; turning over the top of the ladder to the other side of the street without moving its foot, it exactly reaches a window when its new position is at right angles to its former one. Required the width of the street and the height of the latter window.

9. A cruiser from a point N.E. of a harbour observes a blockade-runner in a direction 15° E. of S., running straight for the harbour with a N.W. course; which it pursues, and overtakes at the mouth of the harbour. Compare the rates of sailing.

10. From a ship sailing due S.E. at the rate of 7 miles per hour, a lighthouse is observed to bear N. 30° E., and after two hours its bearing is due N.; find the distance of the ship from the lighthouse at each observation.

11. A privateer descries a barque 12 miles off to the N.E. sailing with a N.W. course, and pursues and overtakes it running due N. for 8 miles, and W. 15° from N. for the remainder of the race. How far did each vessel run in the pursuit?

12. From the top of a ship's mast, 125 feet above its hull, the angle of depression of the hull of another ship is found to be 6° 50′ 34″. Find the distance between the ships.

Given $\tan 6° 50' 34'' = \cdot 12$.

13. A kite has attained an altitude of 36°, and 200 yards of string are out; required the perpendicular height of the kite, supposing the string to be perfectly stretched.

Given $\sin 36° = \cdot 5878$.

14. A castle wall, 20 feet high, is surrounded by a ditch; from the edge of which the altitude of the top of the wall is 60°. Find the width of the ditch, and also how far an observer must walk directly backward in order that the altitude of the wall may be 30°.

Given $\sqrt{3} = 1\cdot 732$.

15. Starting from a point at which I perceive a steeple due E., and a tower due W., I travel for 5 miles and then observe the steeple to be S.E., and the tower to be 15° from S. How far is it from tower to steeple?

Given $\cot 75° = \cdot 26$ nearly.

16. From the top of a hill the angles of depression of two con-

secutive milestones on a straight, level road, were found to be $12° 13'$ and $2° 45'$: find the height of the hill.

Given $\tan 2° 45' = \cdot 048$, $\tan 12° 13' = \cdot 217$.

17. From the edge of one bank of a river a person ascends 100 yards up a slope of 1 in $4\frac{1}{4}$, and observes the angle of depression of an object on the opposite bank close to the edge of the river to be $1\frac{3}{4}$. Find the breadth of the river.

Given $\tan 1° 45' = \cdot 03$.

18. Solve the right-angled triangle ABC, given $a+b$ and A.

19. Also solve the triangle, given $a-b$ and A.

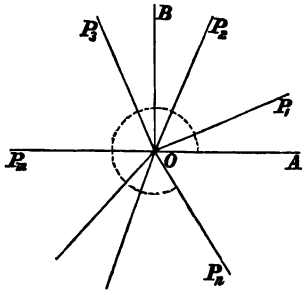
20. Find a, given $b-c$ and A.

21. Solve, given $a+b+c$ and A.

CHAPTER VI.

THE TRIGONOMETRICAL ANGLE AND ITS FUNCTIONS IN THE FOUR QUADRANTS.

35. *The Trigonometrical and Geometrical angle not identical.*



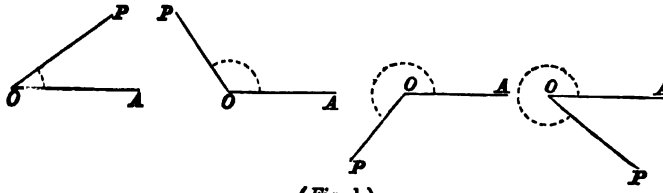
Let OP, starting from the position OA, revolve about O, and assume successively the positions, OP_1, OP_2, OP_3, ... OP_m; where OP_m is in the same straight line with OA. Draw OB at right angles to OA. The angular spaces revolved through by OP are measured successively by the angles AOP_1, AOP_2, &c., until it arrives at the position OP_m. We know it has then revolved through two right angles AOB and BOP_m; and we should infer, from what we have just seen, that AOP_m is an angle equal to two right angles. But AOP_m is not an angle at all by the geometrical definition (Euc. I. def. 9). In our present subject, however, the definition of an angle which is made to suffice for the purposes of Geometry would lead to much inconvenience; and in consequence, it is "extended," as will now be seen.

36. *Extension of the definition of an angle.*

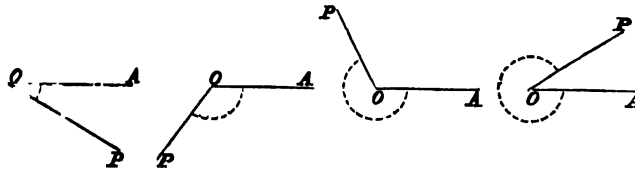
In Trigonometry it is convenient to imagine all our angles traced out, as above, by the turning of a line

about a point from one position to another. Assuming that Geometry furnishes us with our conception of "angular space," we may then define, as follows, an

Angle in Trigonometry. OA being the first position of OP, the angle AOP is the angular space revolved through by OP in coming to its present position.



(*Fig.* 1.)



(*Fig.* 2.)

OP may of course revolve in either of two directions, upwards from OA, or downwards, and form angles of all magnitudes, either as in Fig. 1, or as in Fig. 2.

Hence, in the present subject, AOP_m (figure to Art. 35) *is* an angle; and is measured by two right angles, or 180°, or π. And continuing the revolution before spoken of until OP again coincides with OA, we meet further with angles of all magnitudes between two and four right angles—between 180° and 360°, or π and 2π. More than this, still continuing the revolution about O, we obtain, consistently with our definition, angles greater than four right angles; and it is evident that there is no limit to the magnitude of the trigonometrical angle.

The revolving line is frequently said to *generate* the angle, and is itself spoken of as the *generating line.*

37. It may be interesting to note that Euclid's defi-

nition of an angle is not always adhered to even in Euclid's Elements. Into the demonstration of Prop. 33, Book VI. two angles are introduced; each of which is *any* multiple of another given angle.

If the geometrical angle P_mOP_n (figure to Art. 35.) be 120°, the trigonometrical angle AOP_n indicated by the dotted arc of a circle

$$= \angle AOP_m + 120° = 180° + 120° = 300°.$$

If the generating line be supposed to have made one complete revolution before taking up its present position, the trigonometrical angle generated will then

$$= 360° + 300° = 660°.$$

After two revolutions it would

$$= 2 \times 360° + 300° = 1020°.$$

As this mode of generating angles is not restricted, the line OA and OP_n, or any other two lines under similar conditions, may bound an infinite number of trigonometrical angles.

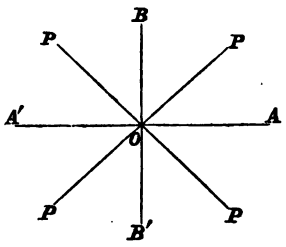
Speaking generally, if A be any angle less than 360°, the expression for all angles bounded by the same two straight lines as A is

$$n \,.\, 360° + A;$$

or, using circular measure, the general expression for all angles bounded by the same straight lines as α is

$$n \,.\, 2\pi + \alpha \quad \text{or} \quad 2n\pi + \alpha.$$

38. *Quadrants.*—Let AOA' and BOB' be at right angles, and OP a line revolving from the position OA.



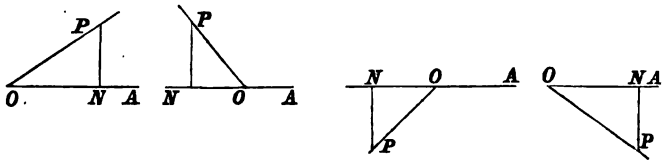
$AOB, BOA', A'OB', B'OA$ are called the *first*, *second*, *third*, and *fourth* quadrants respectively. The angle AOP is said to be in that quadrant in which OP lies.

39. *The Trigonometrical Ratios of Angles generally.*

It was found convenient in a previous chapter to treat

of these ratios in a particular case. We are now in a position to give their definitions; and the statements of Art. 13 may now with some advantage be verified.

DEFINITIONS.



If the straight line OP revolve from the position OA about the centre O, and form the angles AOP of all magnitudes, and if PN be drawn from any point P in OP perpendicular to OA or AO produced; then, writing A for the angle AOP,

$$\sin A = \frac{PN}{OP}, \quad \cos A = \frac{ON}{OP}, \quad \tan A = \frac{PN}{ON},$$

$$\operatorname{cosec} A = \frac{OP}{PN}, \quad \sec A = \frac{OP}{ON}, \quad \cot A = \frac{ON}{PN},$$

$$\operatorname{vers} A = 1 - \cos A.$$

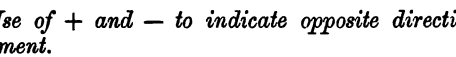
40. It may be useful to some to state here that usages which owe their origin to motives of convenience, and not to those of necessity, are commonly called *conventions*. There are many such usages in Mathematics.

It is now necessary to notice an important and most useful convention; by which, in certain cases, the signs + and − are used to distinguish between opposite directions; and which applies to the above definitions. To discuss its origin and advantages with the fulness they deserve, and indeed properly require, would occupy more space than is here at liberty. As much explanation of the principle, however, as is usually found in treatises on Trigonometry will be given; after which it will be fully described in so far as it applies to the subject in hand.

41. *Use of + and — to indicate opposite directions of measurement.*

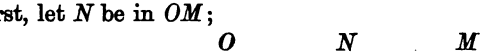
Let MNO be in one straight line, and $OM = a$, $NM = b$, $ON = x$.

First, let N be in OM;



then $$a - b = x.$$

Next, let N move along MO to the left of O;



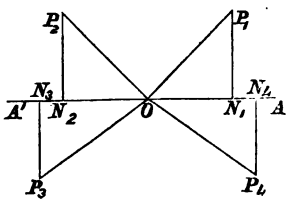
then $$b - a = x,$$
while, therefore, $$a - b = -x.$$

Hence, in order that the same expression $a - b$ may give us the distance of N from O in both the cases we have considered, if ON be measured by a positive symbol when N is to the right of O, it must be measured by a negative symbol when N is to the left of O.

Such a proceeding as this is found of immense utility in innumerable instances in Mathematics, of which the above is a single one. It is therefore adopted as the general mode of proceeding in all those cases: whence the convention of signs; by which, when it is possible to measure magnitudes "in either of two opposite directions, the signs + and — are used to distinguish between the directions."

42. *The Convention of Positive and Negative as applied in Trigonometry—*

(i.) *To certain straight lines.*



Let AA' be a horizontal straight line, and O any point in it. Then the numbers or symbols which measure either perpendiculars to AOA' from points above it, or distances from O to the right hand along OA, are considered *positive;* while those which measure

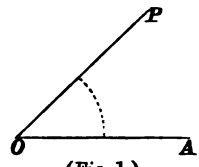
perpendiculars to AOA' from points below it, or distances from O to the left along OA', are considered *negative*.

Thus P_1N_1, P_2N_2, P_3N_3, P_4N_4 being perpendiculars to AOA', P_1N_1, P_2N_2, ON_1, ON_4, are denoted by positive numbers or symbols, such as 2, 5, a, b, x, y, &c.; and P_3N_3, P_4N_4, ON_2, ON_3 by negative numbers or symbols, such as -2, -5, $-a$, $-b$, $-x$, $-y$, &c.

Numbers or symbols representing the revolving line, which traces the trigonometrical angle, are always considered to be positive, *whatever be its position.*

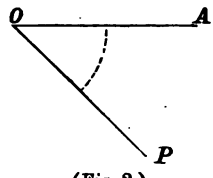
(ii.) *To Angles.*

Again suppose OP to turn about O as a centre. Then the number or symbol which measures the angle AOP is considered



(*Fig.* 1.)

Positive, when, in forming that angle, OP revolves upwards from OA, and moves about O in a direction contrary to that of the hands of a watch, as indicated by the dotted arc in Fig. 1, and also as in Fig. 1, Art. 36;



(*Fig.* 2.)

Negative, when, in forming the angle, OP revolves downwards from OA, and in the direction in which the hands of a watch move, as in Fig. 2, or as in Fig. 2, Art. 36.

For instance, if in each case an angular space of 60° has been revolved through; then the angle AOP in Fig. 1, would be written $\frac{\pi}{3}$ or 60°, while in Fig. 2. it would be written $-\frac{\pi}{3}$ or $-60°$.

EXERCISE III.—The foregoing articles of this Chapter will require very careful attention, and an exercise is therefore appended to them; by carefully working which, the difficulties of the remainder of the Chapter may be materially lessened.

1. What is the magnitude of the angle AOP in figure to Art. 38, when OP bisects $A'OB'$ and has already made one complete revolution?

2. What is its magnitude when OP coincides with OB after making three revolutions?

3. Enumerate the least six trigonometrical angles contained by OA, OB in figures to Art. 38.

4. In which quadrants are the angles: $50°$, $250°$, $-450°$, $650°$, $-850°$, $-1000°$, $\frac{5\pi}{3}$, $2n\pi+\frac{5\pi}{6}$, $-\frac{4\pi}{3}$, $-\left\{(2n+1)\pi+\frac{2\pi}{3}\right\}$?

5. In figure to Art. 42, if P_2N_2 be four, ON_2 three, and OP_2 five linear units in length, write out the trigonometrical functions of the angles AOP_2, affixing appropriate signs to each.

6. In same figure write out the functions of the angle AOP_3, when P_3N_3, ON_3, and OP_3 are respectively y, x, and r linear units in length.

43. The formulæ of Chapter II. have been proved only in the case of angles in the first quadrant. It remains now to be shown how they may be established for angles generally. This will be done by fully stating the proof in the case of the formula

$$\tan A = \frac{\sin A}{\cos A}.$$

Referring to the figure of Art. 39, let OP revolve from OA, forming the angle AOP equal to A; and draw PN perpendicular to OA or AO produced. Suppose PN, ON, and OP to contain a, b, c linear units respectively.

When A is in the *first* quadrant (first figure), the formula is true by Chapter II. We may also show it thus,—

$$\frac{\sin A}{\cos A} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{b} = \tan A.$$

When OP, and consequently A, is in the *second* quadrant (second figure),

$$\frac{\sin A}{\cos A} = \frac{\frac{a}{c}}{\frac{-b}{c}} = \frac{a}{-b} = \tan A.$$

When A is in the *third* quadrant (third figure),

$$\frac{\sin A}{\cos A} = \frac{\frac{-a}{c}}{\frac{-b}{c}} = \frac{-a}{-b} = \tan A.$$

When B is in the *fourth* quadrant (fourth figure),

$$\frac{\sin A}{\cos A} = \frac{\frac{-a}{c}}{\frac{b}{c}} = \frac{-a}{b} = \tan A.$$

Wherefore, for all magnitudes of A,

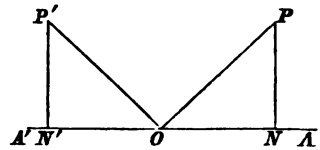
$$\tan A = \frac{\sin A}{\cos A}.$$

44. *Def.*—When the sum of two angles is two right angles, each is called the *supplement* of the other.

That is, A and $180° - A$ are supplements to each other, or are supplementary.

We have already expressed the functions of an angle in terms of those of its complement (Chap. III.). We shall similarly treat the supplement of an angle.

45. *Functions of A in terms of those of* $180° - A$.



Suppose OP to revolve from OA through any angle AOP; and afterwards move into the position OP', such that AO being produced to A', $\angle A'OP' = \angle AOP$. Draw PN and $P'N'$ perpendiculars to AOA'.

Let $AOP = A$, then $AOP' = 180° - A'OP' = 180° - A$; and let PN, ON, and OP contain a, b, c linear units respectively.

Since OP and OP' are equal, $P'N'$ is easily shown equal to PN, and is on the same side of AOA';

$$\therefore \sin(180° - A) = \frac{P'N'}{OP'} = \frac{a}{c}, \text{ and } \sin A = \frac{PN}{OP} = \frac{a}{c};$$

$$\therefore \sin(180° - A) = \sin A, \text{ or } \sin A = \sin(180° - A).$$

Again, ON' is equal to ON, but is to the left of O;

$$\therefore \cos(180^\circ - A) = \frac{ON'}{OP'} = \frac{-b}{c} = -\frac{b}{c},$$

and
$$\cos A = \frac{ON}{OP} = \frac{b}{c};$$

$\therefore \cos(180^\circ - A) = -\cos A$, or $\cos A = -\cos(180^\circ - A)$.

Further,

$$\because \tan(180^\circ - A) = \frac{P'N'}{ON'} = \frac{a}{-b} = -\frac{a}{b},$$

$\therefore \tan(180^\circ - A) = -\tan A$, or $\tan A = -\tan(180^\circ - A)$.

Or thus, as is practically more convenient in the case of all the functions except the sine and cosine,

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin(180^\circ - A)}{-\cos(180^\circ - A)} = -\tan(180^\circ - A).$$

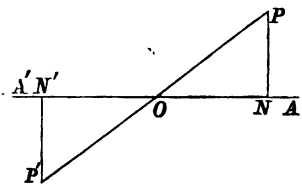
Also

$$\cot A = \frac{\cos A}{\sin A} = \frac{-\cos(180^\circ - A)}{\sin(180^\circ - A)} = -\cot(180^\circ - A),$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{-\cos(180^\circ - A)} = -\sec(180^\circ - A),$$

$$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{1}{\sin(180^\circ - A)} = \operatorname{cosec}(180^\circ - A).$$

46. *Functions of* $180^\circ + A$ *in terms of those of* A.



Suppose OP to revolve from OA through any angle AOP; and afterwards move into the position OP', such that OP' is in the same straight line with OP, and therefore AO being produced to A', $\angle A'OP' = \angle AOP$. Draw PN and $P'N'$ perpendiculars to AOA'.

Let $AOP = A$, then $AOP' = 180^\circ + A'OP' = 180^\circ + A$; and let PN, ON, and OP contain a, b, c linear units respectively.

Then since $P'N'$ is equal to PN, but not on the same side of AOA',

$$\sin(180^\circ + A) = \frac{P'N'}{OP'} = \frac{-a}{c} = -\frac{a}{c},$$

and $$\sin A = \frac{PN}{OP} = \frac{a}{c};$$

$\therefore \sin(180^\circ + A) = -\sin A$, or $\sin A = -\sin(180^\circ + A)$.

Also since ON' is equal to ON, but is to the left of O,

$$\cos(180^\circ + A) = \frac{ON'}{OP'} = \frac{-b}{c} = -\frac{b}{c},$$

and $$\cos A = \frac{ON}{OP} = \frac{b}{c};$$

$\therefore \cos(180^\circ + A) = -\cos A$, or $\cos A = -\cos(180^\circ + A)$.

From formulæ for the sine and cosine may as before be derived those for tangent, &c.

47. *Functions of A in terms of those of $360^\circ - A$.*

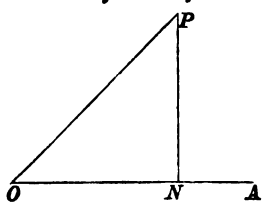
A geometric proof might also be given of this proposition; but it will be easier, and also exhibit an application of the two preceding ones, to proceed as follows:—

$$\sin(360^\circ - A) = \sin\{180^\circ + (180^\circ - A)\}$$
$$= -\sin(180^\circ - A) = -\sin A;$$
$$\therefore \sin A = -\sin(360^\circ - A).$$

Again, $$\cos(360^\circ - A) = \cos\{180^\circ + (180^\circ - A)\}$$
$$= -\cos(180^\circ - A) = \cos A;$$
$$\therefore \cos A = \cos(360^\circ - A).$$

Similarly for the remaining functions.

48. *Functions of $n \,.\, 360^\circ + A$ in terms of those of A.*



Let OP revolve from OA and form any angle AOP; which call A. Afterwards suppose OP to move further and make any number of complete revolutions about O. Its position will be then exactly as before; while, if n

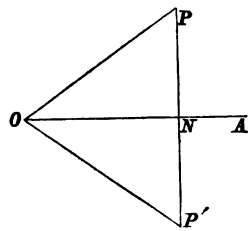
represent the number of revolutions, the angle traced out will be $n \,.\, 360° + A$.

Hence, since the magnitudes and signs of all the functions are of course the same for the same position of OP, the functions of A and of $n \,.\, 360° + A$ are identical; that is,

$$\sin(n \,.\, 360° + A) = \sin A, \quad \cos(n \,.\, 360° + A) = \cos A,$$

and so on.

49. *Functions of* $(-A)$ *in terms of those of* A.



First suppose a line to revolve upwards from OA through the angle AOP; next suppose a line, equal to the former, to revolve downwards from OA through the angle AOP' equal to AOP. Draw perpendiculars from P and P' on OA or AO produced, they will meet in N.

Let AOP be A, then AOP' will be $(-A)$; and let PN, ON, and OP contain a, b, c linear units respectively.

PN is equal to $P'N$, but is not on same side of AO;

$$\therefore \sin(-A) = \frac{P'N}{OP} = \frac{-a}{c} = -\frac{a}{c}, \text{ and } \sin A = \frac{PN}{OP} = \frac{a}{c};$$

$$\therefore \sin(-A) = -\sin A, \quad \text{or} \quad \sin A = -\sin(-A).$$

Again, since

$$\cos(-A) = \frac{ON}{OP'} = \frac{b}{c}, \quad \text{and} \quad \cos A = \frac{ON}{OP} = \frac{b}{c};$$

$$\therefore \cos(-A) = \cos A, \quad \text{or} \quad \cos A = \cos(-A).$$

Hence follow similar formulæ for the other functions.

50. It will serve our purpose here to collect all the formulæ of the above class, which it is desirable should be well and accurately known. The following is not the most symmetrical arrangement possible; but it will

probably be found the best to remember for practical purposes.

$\sin A = \cos(90^\circ - A)$, $\cos A = \sin(90^\circ - A)$.....(1);
$\sin A = \sin(180^\circ - A)$, $\cos A = -\cos(180^\circ - A)$...(2);
$\sin(180^\circ + A) = -\sin A$, $\cos(180 + A) = -\cos A$...(3);
$\sin A = -\sin(360^\circ - A)$, $\cos A = \cos(360^\circ - A)$ ..(4);
$\sin(n.360^\circ + A) = \sin A$, $\cos(n.360^\circ + A) = \cos A$...(5);
$\sin(-A) = -\sin A$, $\cos(-A) = \cos A$ .....(6).

It will be clear enough by this time, how the corresponding formulæ for the tangents, cotangents, secants, and cosecants are very readily deduced from the above; so that it is not necessary to burden the memory with them.

51. *To express a function of any angle greater than* 90° *in terms of the same function of an angle equal to or less than* 90°.

For this purpose the above formulæ are adapted. For

(i.) If the angle be in the second, third, or fourth quadrants, evidently (2), (3), or (4), will respectively apply. Thus,

by (2), $\sin 120^\circ = \sin(180^\circ - 120^\circ) = \sin 60^\circ$;
by (3), $\cos 220^\circ = \cos(180^\circ + 40^\circ) = -\cos 40^\circ$;
by (4), $\tan 340^\circ = -\tan(360^\circ - 340^\circ) = -\tan 20^\circ$.

(ii.) If the angle be greater than 360°, (5) will reduce it; and if it be still greater than 90°, either (2), (3), or (4) will then apply. Thus,

$$\cot 440^\circ = \cot(360^\circ + 80^\circ) = \cot 80^\circ; \text{ and}$$
$$\sec 1050^\circ = \sec(2 . 360^\circ + 330^\circ) = \sec 330^\circ$$
$$= \sec(360^\circ - 330^\circ) = \sec 30^\circ.$$

(iii.) If the angle be negative, (6) will render it positive; and if it be then greater than 90°, either the methods of (ii.) or (i.) will reduce. Thus,

$$\operatorname{cosec}(-45^\circ) = -\operatorname{cosec} 45^\circ; \text{ and}$$
$$\sin\left(-\frac{5\pi}{2}\right) = -\sin\frac{5\pi}{2} = -\sin\left(2\pi + \frac{\pi}{2}\right) = -\sin\frac{\pi}{2}.$$

In instances such as this last, the converse of (5) sometimes applies most successfully. Thus,

$$\sin\left(-\frac{7\pi}{2}\right) = \sin\left(2 \,.\, 2\pi - \frac{7\pi}{2}\right) = \sin\frac{\pi}{2}.$$

52. (1.) Find the value of $\tan 150°$ and of $\cos\left\{(4n+1)\frac{\pi}{2}+\frac{\pi}{3}\right\}$.

$$\tan 150° = -\tan(180° - 150°) = -\tan 30°;$$

$$\therefore\ \tan 150° = -\frac{1}{\sqrt{3}}.$$

And $\cos\left\{(4n+1)\frac{\pi}{2}+\frac{\pi}{3}\right\} = \cos\left\{2n\pi + \left(\frac{\pi}{2}+\frac{\pi}{3}\right)\right\}$

$$= \cos\frac{5\pi}{6} = -\cos\left(\pi - \frac{5\pi}{6}\right) = -\cos\frac{\pi}{6};$$

$$\therefore\ \cos\left\{(4n+1)\frac{\pi}{2}+\frac{\pi}{3}\right\} = -\frac{\sqrt{3}}{2}.$$

(2.) Find the values of A less than $360°$ which satisfy the equation,

$$\cot A + 2\cos A = 0.$$

Transposing, &c., in given equation,

$$2\cos A = -\frac{\cos A}{\sin A};$$

$$\therefore\ 2 = -\frac{1}{\sin A},$$

$$\therefore\ \sin A = -\frac{1}{2}$$

$$= -\sin 30° = \sin 210°, \text{ or } \sin 330°;$$

$$\therefore\ A = 210°, \text{ or } 330°.$$

Examples.

1. Find expressions, in which the angles are less than $90°$, equivalent to the following:—

$$\cos 160°,\quad \tan 160°,\quad \sec(-200°),\quad \operatorname{cosec}(-1000°).$$

2. Prove that,

$$\cos(90° + A) = -\sin A, \text{ and } \sin(90° + A) = \cos A.$$

3. Simplify the expressions:—

$$\sin\left(\frac{3\pi}{2} - a\right),\quad \tan\left(\frac{3\pi}{2} - a\right),\quad \cot\left(\frac{3\pi}{2} + \theta\right),\quad \sec\left(\frac{3\pi}{2} + \theta\right).$$

4. Find the values of $\cos 135°$, $\cot 225°$, and $\operatorname{cosec}(-510°)$.

5. Find the values of $\sin\left(2n\pi + \frac{2\pi}{3}\right)$, $\cos\left\{-\left(2n\pi - \frac{\pi}{6}\right)\right\}$, $\tan\left\{(2n+1)\pi - \frac{\pi}{4}\right\}$, and $\cot\left\{(4n+3)\frac{\pi}{2} - \frac{5\pi}{6}\right\}$.

6. What values of A less than 540° satisfy the equation

$$\cos A = -\frac{\sqrt{3}}{2}?$$

7. What are the least eight angles which satisfy the equation

$$\cot^2\theta = 1?$$

8. Prove geometrically, for all values of A, that

$$\cos A = \frac{1}{\sec A}.$$

9. Prove geometrically, for angles generally, the formula

$$\sin^2 A + \cos^2 A = 1.$$

10. Find all the values of A less than 360° satisfying the equation

$$2\sin^2 A + \sin A = 1.$$

11. Find all the angles less than 360° satisfying the equation

$$2\sin^2 A = 3\cos A.$$

12. Find values of θ less than π satisfying the equation

$$\tan^2\theta + \cot^2\theta = \frac{10}{3}.$$

13. For what angle is the supplement equal to the sine of the complement?

14. Write down the different values of $\cos\left(\frac{n\pi}{2} - A\right)$, when n is 0, 1, 2, 3, &c.

15. Prove geometrically the formulæ

$$\sin\left(\frac{\pi}{2} + A\right) = \cos A, \quad \text{and} \quad \cos\left(\frac{\pi}{2} + A\right) = -\sin A.$$

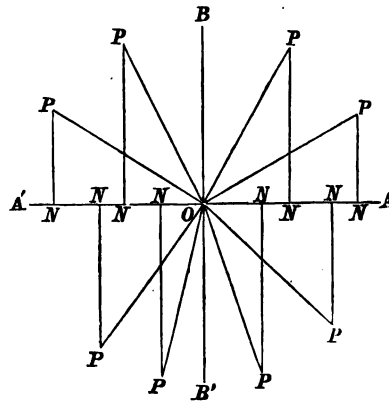
CHAPTER VII.

THE VARIATIONS IN MAGNITUDE AND SIGN OF THE TRIGONOMETRICAL RATIOS.

53. In order that not even the lesser difficulties of the subject may be altogether passed over, the following proposition is more fully stated than those which follow it, which are of a similar nature.

54. *To trace the variation in sign and magnitude of sin A, as A increases from* 0 to 360°.

Let AOA' and BOB' be at right angles; and suppose OP to revolve from OA through all the four quadrants. Draw PN perpendicular to AOA'.



Also, let A be the angle AOP, y the linear units in PN, and r the linear units in OP, for all positions of OP. Then

$$\sin A = \frac{y}{r}.$$

Since the symbol representing OP is always positive and does not vary in value, the changes in the value of sin A will all depend upon those of y.

Whilst OP coincides with OA, y is evidently 0; therefore when $A = 0$, $\sin A = \frac{0}{r}$; $\therefore \sin 0 = 0$.

As A increases from 0 to 90°, y increases and is positive; therefore $\sin A$ increases and is positive.

When OP arrives at the position OB, PN coincides with OP, and y is therefore equal to r in magnitude; it is positive in sign; therefore

$$\text{when } A = 90°, \sin A = \frac{r}{r}; \therefore \sin 90° = 1.$$

As A increases from 90° to 180°, y decreases and is positive; therefore $\sin A$ decreases and is positive.

When OP arrives at the position OA', PN vanishes; therefore

$$\text{when } A = 180°, \sin A = \frac{0}{r}; \therefore \sin 180° = 0.$$

As A increases from 180° to 270°, y increases and is negative; therefore $\sin A$ increases and is negative.

When OP comes to OB', PN again coincides with OP; and y is therefore equal to r in magnitude, but its sign is negative; hence

$$\text{when } A = 270°, \sin A = \frac{-r}{r}; \therefore \sin 270° = -1.$$

As A increases from 270° to 360°, y decreases and is negative; therefore $\sin A$ decreases and is negative.

When OP returns to its first position OA, PN again vanishes; therefore

$$\text{when } A = 360°, \sin A = \frac{0}{r}; \therefore \sin 360° = 0.$$

Summary.—Let angle A pass continuously through all values from 0 to 360°; then

When A is	0 or 360°	In 1st Quadrant.	90°	In 2nd Quadrant.	180°	In 3rd Quadrant	270°	In 4th Quadrant.
$\sin A$ is	0	+ and increasing.	1	+ and decreasing.	0	− and increasing.	−1	− and decreasing.

55. *To trace the variation in sign and magnitude of cos A, as A increases from* 0 *to* 360°.

With the construction of last proposition, let A be the angle AOP, x the linear units in ON, and r the linear units in OP for all positions of OP. Then

$$\cos A = \frac{x}{r}.$$

When $A = 0$, ON coincides with OP, and

$$\cos A = \frac{r}{r}; \therefore \cos 0 = 1.$$

As A increases from 0 to 90°, x decreases and is positive; therefore cos A decreases and is positive.

When $A = 90°$, ON vanishes, and

$$\cos A = \frac{0}{r}; \therefore \cos 90° = 0.$$

As A increases from 90° to 180°, x increases and is negative; therefore cos A increases and is negative.

When $A = 180°$, ON coincides with OP, and

$$\cos A = \frac{-r}{r}; \therefore \cos 180° = -1.$$

As A increases from 180° to 270°, x decreases and is negative; therefore cos A decreases and is negative.

When $A = 270°$, ON vanishes, and

$$\cos A = \frac{0}{r}; \therefore \cos 270° = 0.$$

As A increases from 270° to 360°, x increases and is positive; therefore cos A increases and is positive.

When $A = 360°$, ON coincides with OP, and

$$\cos A = \frac{r}{r}; \therefore \cos 360° = 1.$$

A tabular *summary* may be constructed as before (Art. 58).

56. *To trace the variation in sign and magnitude of tan A, as A increases from* 0 *to* 360°.

With the same construction as before, let A be angle

AOP, y the linear units in PN, and x the linear units in ON for all positions of OP. Then

$$\tan A = \frac{y}{x}.$$

When $A = 0$, PN vanishes, and ON coincides with OP; hence

$$\tan A = \frac{0}{r}; \therefore \tan 0 = 0.$$

As A increases from 0 to 90°, y increases and x decreases, while both are positive; therefore $\tan A$ increases and is positive.

When $A = 90°$, PN coincides with OP, and ON vanishes; hence

$$\tan A = \frac{r}{0}, \therefore \tan 90° = \infty.$$

As A increases from 90° to 180°, y decreases and is positive, while x increases and is negative; therefore $\tan A$ decreases and is negative.

When $A = 180°$, PN vanishes, and ON coincides with OP; hence

$$\tan A = \frac{0}{-r}; \therefore \tan 180° = 0.$$

As A increases from 180° to 270°, y increases and x decreases, while both are negative; therefore $\tan A$ increases and is positive.

When $A = 270°$; PN coincides with OP, and ON vanishes; hence

$$\tan A = \frac{-r}{0}; \therefore \tan 270° = \infty.$$

As A increases from 270° to 360°, y decreases and is negative, while x increases and is positive; therefore $\tan A$ decreases and is negative.

When $A = 360°$, PN vanishes, and ON coincides with OP; hence

$$\tan A = \frac{0}{r}; \tan 360° = 0.$$

A tabular *summary* may, as before, be constructed.

57. We have now virtually shown how also to trace the variations in magnitude and sign of the cosecant, secant, and cotangent of an angle as the angle varies; for these are easily deducible from those of the sine, cosine, and tangent respectively. For example, we may treat the secant thus:—

$$\sec 0 = \frac{1}{\cos 0} = \frac{1}{1}; \therefore \sec 0 = 1.$$

As A increases from 0 to 90°, cos A decreases and is positive; therefore sec A or $\frac{1}{\cos A}$ increases and is positive.

$$\sec 90^\circ = \frac{1}{\cos 90^\circ} = \frac{1}{0}; \therefore \sec 90^\circ = \infty.$$

As A increases from 90° to 180°, cos A increases and is negative; therefore sec A decreases and is negative.

And so on throughout the four quadrants.

Again, the same variations for the cosecant, secant, and cotangent may be traced by means of a figure, exactly as those for the sine, cosine, and tangent have been traced.

58. *General summary.*—Let angle A pass continuously through all values from 0 to 360°; then

When A is	0 or 360°	In 1st Quadrant.	90°	In 2nd Quadrant.	180°	In 3rd Quadrant.	270°	In 4th Quadrant.
sin A is	0	+ and increasing.	1	+ and decreasing.	0	− and increasing.	−1	− and decreasing.
cos A is	1	+ and decreasing.	0	− and increasing.	−1	− and decreasing.	0	+ and increasing.
tan A is	0	+ and increasing.	∞	− and decreasing.	0	+ and increasing.	∞	− and decreasing.
cot A is	∞	+ and decreasing.	0	− and increasing.	∞	+ and decreasing.	0	− and increasing.
sec A is	1	+ and increasing.	∞	− and decreasing.	−1	− and increasing.	∞	+ and decreasing.
cosec A is	∞	+ and decreasing.	1	+ and increasing.	∞	− and decreasing.	−1	− and increasing.

59. It should be observed that 0 and ∞ are *points of transition* from positive to negative and

from negative to positive for continuously varying quantities. Thus the preceding investigations show that

sines and cosines change sign on passing through 0,
tangents and cotangents on passing through 0 and ∞,
and *secants and cosecants* on passing through ∞.

Again, it is interesting to notice the limits of the trigonometrical ratios, and the manner in which each of them varies from one of its limits to the other.

The limits of the sine are 1 and -1; and it moves from the first to the second by decreasing, passing through 0 and changing sign, and increasing numerically. The limits of the cosecant are the same; but it may be said to pass from one to the other by a different route: it increases, passes through ∞ and changes sign, and decreases numerically. What has been said of sine and cosecant applies exactly to cosine and secant.

The values of the tangent and cotangent have no limits. All values positive or negative are possible to them.

60. (1.) What values of θ less than 2π satisfy the equation

$$\cot^2\theta = -\cot\theta.$$

Transposing, &c., in given equation,

$$\cot\theta\,(\cot\theta + 1) = 0.$$

Therefore (i.) ... $\cot\theta = 0 = \cot\frac{\pi}{2}$ or $\cot\frac{3\pi}{2}$:

and (ii.) ..... $\cot\theta + 1 = 0$;

$$\therefore \cot\theta = -1 = \cot\frac{3\pi}{4} \text{ or } \cot\frac{7\pi}{4}.$$

Hence required values of θ are $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{3\pi}{4}, \frac{7\pi}{4}$.

(2.) Trace the changes in $2\cos\frac{\theta}{2}$ as θ varies from 0 to 2π.

When $\theta = 0$, $2\cos\frac{\theta}{2} = 2\cos 0 = 2$.

As θ increases from 0 to π, $\frac{\theta}{2}$ increases from 0 to $\frac{\pi}{2}$; therefore $2\cos\frac{\theta}{2}$ decreases and is positive.

When $\theta = \pi$, $2 \cos \frac{\theta}{2} = 2 \cos \frac{\pi}{2} = 0$.

As θ increases from π to 2π, $\frac{\theta}{2}$ increases from $\frac{\pi}{2}$ to π; therefore $2 \cos \frac{\theta}{2}$ increases and is negative.

When $\theta = 2\pi$, $2 \cos \frac{\theta}{2} = 2 \cos \pi = -2$.

(3.) Trace the variation in sign of $\sin A - \cos A$, as A increases from 0 to 360°.

When A is between 0 and 45°, $\cos A$ is greater than $\sin A$ and is positive; therefore $\sin A - \cos A$ is *negative*.

When A is between 45° and 135°, $\sin A$ is greater than $\cos A$ and positive; therefore $\sin A - \cos A$ is *positive*.

When A is between 135° and 225°, $\cos A$ is greater than $\sin A$ and negative; therefore $\sin A - \cos A$ is *positive*.

When A is between 225° and 315°, $\sin A$ is greater than $\cos A$ and negative; therefore $\sin A - \cos A$ is *negative*.

When A is between 315° and 360°, $\cos A$ is greater than $\sin A$ and positive, therefore $\sin A - \cos A$ is *negative*.

Examples.

1. Trace by means of a figure the variation in magnitude and sign of $\cot A$, as A increases from 0 to 360°.

2. Prove geometrically that when A is less than 45°, $\sin A$ is less than $\cos A$; and when A is between 45° and 135°, $\sin A$ is greater than $\cos A$.

3. Show geometrically which is greater, $\tan A$ or $\cot A$, when A is between 135° and 225°.

4. Show geometrically the absurdity of the equations

$$\sin A = \tfrac{3}{5}, \quad \sec A = \cdot 8.$$

5. A_1 is in the first quadrant, and A_2 in second quadrant, while both are very near to 90° in magnitude: compare $\tan A_1$ and $\tan A_2$; also $\sec A_1$ and $\sec A_2$.

6. For what values of A less than 360° is $\cot A$ very great in magnitude and negative in sign?

7. For what values of A less than 360° is $\operatorname{cosec} A$ a very large negative quantity?

8. Deduce from a geometrical construction the range of the values of $\sec A$, A being any angle.

9. What are the nearest approaches of the cosecant to zero in value?

10. Show that the tangent may have any finite value, positive or negative. Is this also true of the cotangent and secant?

Solve the following equations for values of θ less than 2π :—

11. $\sin^2\theta = 1$.

12. $\cos^2\theta + \cos\theta = 2$.

13. $\sec^2\theta - \sec\theta = 2$.

14. $\tan^2\theta + \sqrt{3}\tan\theta = 0$.

15. $\sqrt{3}\operatorname{cosec}^2\theta - (2+\sqrt{3})\operatorname{cosec}\theta + 2 = 0$.

Trace the changes in sign and value of the following expressions, as θ changes continuously from θ to 2π :—

16. $\operatorname{versin}\theta$.

17. $2\operatorname{cosec}\frac{\theta}{2}$.

18. $\sec^2\theta$.

19. $\sin\theta + \cos\theta$.

20. $\cot^2\theta - \tan^2\theta$.

CHAPTER VIII.

GENERAL EXPRESSIONS FOR ANGLES HAVING THE SAME SINE, COSINE, &c.

61. If not already familiar with them, the student should notice the following points in Algebra.

We commonly use n to represent any whole number whatever. Whenever such is its meaning, since the doubles of all numbers are necessarily even, all expressions like the following,

$$2(n-1),\ 2n,\ 2(n+1),\ \&c.,$$

clearly represent *even* numbers.

On very little reflection it will appear that adding an odd number to or subtracting an odd number from an even number produces an odd number. Then expressions like the following—

$$2n-1,\ 2n+1,\ 2n+3,\ \&c.$$

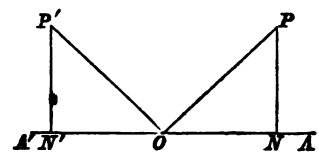
must represent *odd* numbers.

Again, the value of the expression $(-1)^n$ is $+1$ when n is 0 or an even number, but -1 when n is an odd number; for

$$(-1)^0=1,\ (-1)^1=-1,\ (-1)^2=1,\ (-1)^3=-1,$$
$$(-1)^4=1,$$

and so on.

62. *To find an expression for all angles which have a given value for sine.*



Make AOP equal to the least positive angle whose sine is the given value; produce AO to A', and make $A'OP'$ equal to AOP. Then it is easily shown that

$$\sin AOP' = \sin AOP;$$

and also that the sine of no other angle less than four right angles is equal to sin AOP.

Evidently, then, *all* the angles contained by OA and OP, and by OA and OP', and no others, will have the given value for sine.

Let $AOP = \alpha$, so that $AOP' = \pi - \alpha$.

Then all the *positive* angles required are contained in the expressions

$$2n\pi + \alpha \text{ and } 2n\pi + (\pi - \alpha),$$

that is, $$2n\pi + \alpha \text{ and } (2n+1)\pi - \alpha. \quad \ldots\ldots\ldots\ldots(1),$$

where n is 0 or any positive integer (Art. 37).

Since AOP' and AOP are the negative angles $-(\pi + \alpha)$ and $-(2\pi - \alpha)$, all the *negative angles* required are contained in the expressions

$$2n\pi - (\pi + \alpha) \text{ and } 2n\pi - (2\pi - \alpha);$$

that is, $$(2n-1)\pi - \alpha \text{ and } 2(n-1)\pi + \alpha. \quad \ldots\ldots\ldots(2),$$

where n is 0 or any negative integer.

In (1) and (2) are four expressions to be replaced by one expression. On inspecting them it is evident that n, considering it any even or odd, positive or negative number, may represent the coefficient of π. The α which follows an even coefficient of π is $+$, while that which follows an odd coefficient of π is $-$; therefore $(-1)^n$ will fix the sign of α. Hence the expression sought is

$$n\pi + (-1)^n \alpha,$$

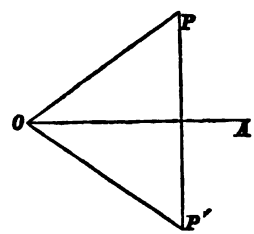
where n is any positive or negative integer.

It is usual to express this result as a formula, thus—

$$\sin \alpha = \sin \{n\pi + (-1)^n \alpha\}.$$

63. *To find an expression for all the angles which have a given value for cosine.*

Make $AOP = \alpha$, the least positive angle whose cosine is the given value; and AOP' geometrically equal to AOP.



It is easily seen that, *all* the trigonometrical angles contained by OA and OP, and by OA and OP', and *no others*, will have the given value for cosine.

Then all the *positive* angles required are contained in the expressions

$$2n\pi+a \text{ and } 2n\pi+(2\pi-a),$$

or, $$2n\pi+a \text{ and } 2(n+1)\pi-a. \quad\ldots\ldots\ldots\ldots(1),$$

where n is 0 or any positive integer.

And all the *negative* angles required are contained in the expressions

$$2n\pi-a \text{ and } 2n\pi-(2\pi-a),$$

or $$2n\pi-a \text{ and } 2(n-1)\pi+a. \quad\ldots\ldots\ldots\ldots(2),$$

where n is 0 or any negative integer (Art. 37).

The coefficient of π in the expressions of (1) and (2) is always even, and both $+a$ and $-a$ occur. These expressions may be therefore replaced by

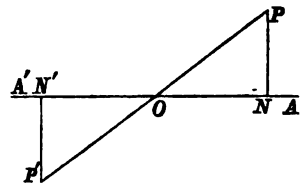
$$2n\pi\pm a,$$

where n is 0 or any positive or negative integer.

Hence the formula

$$\cos a = \cos(2n\pi\pm a).$$

64. *To find an expression for all angles which have a given value for tangent.*



Make $AOP = a$, the least positive angle whose tangent is the given value; and produce AO to A' and PO to P'.

It is easily shown that all the trigonometrical angles contained by OA and OP, and by OA and OP', and no others, have the given value for tangent.

Then all the positive angles required are contained in the expressions

$$2n\pi+a \text{ and } 2n\pi+(\pi+a),$$

or $$2n\pi+a \text{ and } (2n+1)\pi+a. \quad\ldots\ldots\ldots\ldots(1),$$

where n is 0 or any positive integer (Art. 37).

And all the negative angles required are contained in the expression

$$2n\pi-(\pi-a) \text{ and } 2n\pi-(2\pi-a),$$

or $$(2n-1)\pi+a \text{ and } 2(n-1)\pi+a. \quad\ldots\ldots\ldots(2),$$

where n is 0 or any negative integer.

The coefficients of π are either even or odd, and α is always $+$ in (1) and (2). We may therefore replace them by

$$n\pi+\alpha,$$

where n is 0 or any positive or negative integer.

Hence $$\tan\alpha = \tan(n\pi+\alpha).$$

65. From considerations already touched upon, we may consider as established

$$\cot\alpha = \cot(n\pi+\alpha),$$
$$\sec\alpha = \sec(2n\pi\pm\alpha),$$

and $$\operatorname{cosec}\alpha = \operatorname{cosec}\alpha\left\{n\pi+(-1)^n\alpha\right\}.$$

66. There will be found in the foregoing Chapters, as well as in those which follow, many instances of the occurrence in general formulæ of the square root sign. To each of these the sign $\pm$ is either attached or understood, indicating ambiguity; and it is possible in all such cases to show that there is real ambiguity so long as the formulæ are perfectly general, and we know nothing of the angles involved. We will illustrate by considering some of the formulæ in Chapter II., which involve radicals.

67. *Ambiguity in the expression for sin A in terms of cos A.*

The first of the formulæ referred to expresses $\sin A$ in terms of $\cos A$, and is

$$\sin A = \pm\sqrt{1-\cos^2 A}.$$

Now in this Chapter we have shown that

$$\cos A = \cos(2n\,.\,180^\circ\pm A);$$

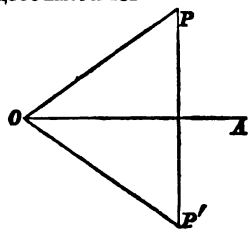
so that $$\pm\sqrt{1-\cos^2 A} = \pm\sqrt{1-\cos^2(2n\,.\,180^\circ\pm A)}.$$

But this last radical is the expression for $\sin(2n\,.\,180^\circ\pm A)$; and the original formula in its most general form should therefore stand thus:

$$\sin(2n\,.\,180^\circ\pm A) = \pm\sqrt{1-\cos^2(2n\,.\,180^\circ\pm A)}.$$

The double sign may now be accounted for

(i.) *Geometrically.*—Make the angles AOP and AOP' each geometrically equal to A.



In the expression $2n.180°\pm A$ there are two groups of trigonometrical angles, one group bounded by the straight lines OA and OP, and the other by OA and OP'.

The sines of those bounded by AO and OP will be positive, while the sines of the others are evidently negative. Hence the value of $\sin(2n.180°\pm A)$ has appropriately a double sign.

(ii.) *Analytically.*—By previous formulæ (Chap. VI.),

$$\sin(2n.180°\pm A) = \sin(n.360°\pm A)$$
$$= \sin(\pm A), = \pm\sin A;$$

from which follows the conclusion before obtained.

68. *Other ambiguities.*—As before stated, there are many such ambiguities. We can only treat here of the simplest of them; and give some indication in that way of the manner in which all others may be considered.

It will now be seen that the most general way of expressing the formula,

$$\cos A = \pm\sqrt{1-\sin^2 A},$$

is $\cos\{n.180°+(-1)^n A\}$

$$= \pm\sqrt{1-\sin^2\{n.180°+(-1)^n A\}},$$

and there is a double sign; for

(1) If n be even, and equal to $2m$ suppose; then

$$\cos\{n.180°+(-1)^n A\} = \cos\{m.360°+A\}$$
$$= \cos A;$$

(2) if n be odd, and equal to $2m+1$ suppose; then

$\cos\{n.180°+(-1)^n A\}$

$$= \cos\{(2m+1)\,180°+(-1)^{2m+1}A\}$$
$$= \cos\{m.360°+(180°-A)\}$$
$$= \cos(180°-A) = -\cos A.$$

Therefore, for both even and odd values of n,

$$\cos\{n.180^\circ+(-1)^n A\} = \pm\cos A;$$

and the double sign is required.

Again, $\sec A = \pm\sqrt{\tan^2 A+1}$;

and $\tan A = \tan(n.180^\circ+A)$;

therefore, $\sec(n.180^\circ+A) = \pm\sqrt{\tan^2(n.180^\circ+A)}$;

and we may show that

$$\sec(n.180^\circ+A) = \pm\sec A,$$

according as n is even or odd.

69. When a particular value is given to the angle involved in such formulæ, or certain limits between which it must lie, the ambiguity vanishes.

For instance (by Chap. VII.), if A be in the first quadrant,

$$\sin A = \sqrt{1-\cos^2 A};\quad \cos A = \sqrt{1-\sin^2 A},$$
$$\tan A = \sqrt{\sec^2 A-1}.$$

If A be in the second quadrant, then

$$\sin A = \sqrt{1-\cos^2 A},\quad \cos A = -\sqrt{1-\sin^2 A},$$
$$\tan A = -\sqrt{\sec^2 A-1}.$$

If A be in the third quadrant, then

$$\sin A = -\sqrt{1-\cos^2 A},\quad \cos A = -\sqrt{1-\sin^2 A},$$
$$\tan A = \sqrt{\sec^2 A-1}.$$

If A be in the fourth quadrant, then

$$\sin A = -\sqrt{1-\cos^2 A},\quad \cos A = \sqrt{1-\sin^2 A},$$
$$\tan A = -\sqrt{\sec^2 A-1}.$$

Similarly, we know the sign of a radical in all such cases.

70. With the aid of the formulæ of this Chapter, we may obtain all the values of angles which satisfy equations in which they are involved.

Solve the equation

$$2\sin^2\theta+5\sin\theta+2 = 0.$$

Proceeding as usual in the solution of such equations, we arrive at the result,

$$\sin\theta = -2 \text{ or } -\tfrac{1}{2}.$$

The value -2 is impossible, for all possible sines lie within the limits 1 and -1; and

$$-\frac{1}{2} = \sin\frac{5\pi}{6} = \sin\left\{n\pi + (-1)^n\frac{5\pi}{6}\right\};$$

$$\therefore\ \theta = n\pi + (-1)^n\frac{5\pi}{6}.$$

EXAMPLES.

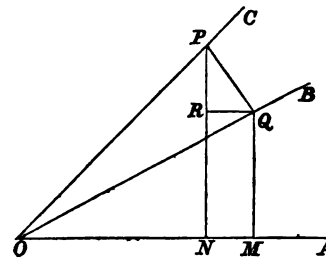
Solve the equations,—

1. $2\tan a = \sqrt{3}\sec a$.
2. $\tan A = 3\cot A$.
3. $\cot A = \sqrt{2}\cos A$.
4. $2\operatorname{cosec}\theta - \cot\theta = \sqrt{3}$.
5. $2\sec^2\theta + (2\sqrt{2}+1)\sec\theta + \sqrt{2} = 0$.
6. $\cot^2\theta + 4\cos^2\theta = 6$.
7. $\sec\theta = -1$.
8. $\sin A + \cos A = 2$.
9. $\cos a = \sec a$.
10. $\cot^2\theta + \sqrt{3}\cot\theta = 0$.
11. Geometrically account for the double sign in
$$\cos A = \pm\sqrt{1-\sin^2 A}.$$
12. Account geometrically and analytically for the ambiguity in the formula,
$$\operatorname{cosec} A = \pm\sqrt{\cot^2 A + 1}.$$

CHAPTER IX.

FUNCTIONS OF SUMS, DIFFERENCES, MULTIPLES, AND SUB-MULTIPLES OF ANGLES.

71. *Given the sines and cosines of any two angles whose sum is less than a right angle, to find the sine and cosine of the sum.*



Let A and B be the two angles:
make $\angle AOB = A$,
and $\angle BOC = B$;
then $A+B = \angle AOC$.

In OC take any point P; and draw PN and and PQ perpendicular to OA and OB respectively, also QM and QR perpendicular to OA and PN; then

$$\angle RPQ = 90° - \angle PQR = \angle OQR = A.$$

$$\text{Now, } \sin(A+B) = \frac{PN}{OP} = \frac{QM+PR}{OP} = \frac{QM}{OP} + \frac{PR}{OP}$$

$$= \frac{QM}{OQ} \cdot \frac{OQ}{OP} + \frac{PR}{PQ} \cdot \frac{PQ}{OP};$$

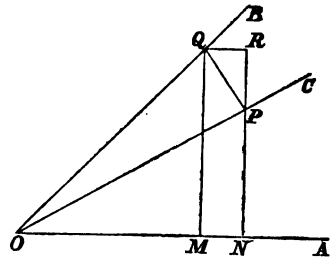
therefore $\sin(A+B) = \sin A \cos B + \cos A \sin B$... (1).

$$\text{Also, } \cos(A+B) = \frac{ON}{OP} = \frac{OM-QR}{OP} = \frac{OM}{OP} - \frac{QR}{OP}$$

$$= \frac{OM}{OQ} \cdot \frac{OQ}{OP} - \frac{QR}{PQ} \cdot \frac{PQ}{OP};$$

therefore $\cos(A+B) = \cos A \cos B - \sin A \sin B$ ...(2).

72. *Given the sines and cosines of any two angles, of which the greatest is less than a right angle: to find the sine and cosine of their difference.*



Let A be the greater and B the less of the two angles:

make $\angle AOB = A$,

and $\angle BOC = B$;

then $A - B = \angle AOC$.

In OC take any point P, and draw PN and PQ perpendicular to OA and OB respectively, also QM and QR perpendicular to OA and PN produced; then

$$\angle RPQ = 90° - \angle PQR = \angle BQR = A.$$

$$\text{Now, } \sin(A-B) = \frac{PN}{OP} = \frac{QM - PR}{OP} = \frac{QM}{OP} - \frac{PR}{OP}$$

$$= \frac{QM}{OQ} \cdot \frac{OQ}{OP} - \frac{PR}{PQ} \cdot \frac{PQ}{OP};$$

therefore $\sin(A-B) = \sin A \cos B - \cos A \sin B$ ... (3).

$$\text{Also, } \cos(A-B) = \frac{ON}{OP} = \frac{OM + QR}{OP} = \frac{OM}{OP} + \frac{QR}{OP};$$

$$= \frac{OM}{OQ} \cdot \frac{OQ}{OP} + \frac{QR}{PQ} \cdot \frac{PQ}{OP};$$

therefore $\cos(A-B) = \cos A \cos B + \sin A \sin B$ ... (4).

73. The formulæ of the two preceding articles are of such importance in the subsequent portions of the subject, as to be usually considered its *fundamental* formulæ. The same geometrical construction as before, with some slight modifications, and a careful adherence to the convention of positive and negative, would suffice to demonstrate them for angles of *all* magnitudes; and when reading the subject for the second time, the student

would find it a most useful exercise to satisfy himself of the fact. The following mode of proof is preferred here, however, because it will probably be found more simple, and certainly requires less space.

74. *The Fundamental Formulæ demonstrated generally.*

The formulæ of Art. 71 are not proved for all values of A and B in the first quadrant, as is the case with those of Art. 72; but they are evidently shown true for any values of A and B less than $45°$. They will now be proved when A and B are between $45°$ and $90°$.

Suppose A and B to be any angles between $45°$ and $90°$ in magnitude, and equal to $90°-A'$ and $90°-B'$ respectively, so that A' and B' are between 0 and $45°$. Then

$$\begin{aligned}\sin(A+B) &= \sin(90°-A'+90°-B')\\ &= \sin\{180°-(A'+B')\} = \sin(A'+B')\\ &= \cos A' \sin B' + \sin A' \cos B'\\ &= \sin(90°-A')\cos(90°-B')\\ &\qquad + \cos(90°-A')\sin(90°-B');\end{aligned}$$

$$\therefore \sin(A+B) = \sin A \cos B + \cos A \sin B.$$

By similar reasoning,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

Hence the fundamental formulæ are *all* true for all angles in the *first* quadrant.

Now, let A be any angle in the *second* quadrant, and equal to $90°+A'$, so that A' is in the first quadrant; take B also in the first quadrant. It has been shown that

$$\sin(90°+A) = \cos A, \text{ and } \cos(90°+A) = -\sin A;$$

$$\begin{aligned}\therefore \sin(A+B) &= \sin(90°+A'+B) = \cos(A'+B)\\ &= \cos A' \cos B - \sin A' \sin B\\ &= \sin(90°+A')\cos B + \cos(90°+A')\sin B,\end{aligned}$$

$$\text{or } \sin(A+B) = \sin A \cos B + \cos A \sin B.$$

The same process applied to this last result will prove the formula when A is in the *third* quadrant; and it is

clear how it may then be established when A is in the *fourth* quadrant. Its second angle (B) may be treated in the same manner.

The formula is therefore true generally for *positive* values of the angles involved.

Next, since

$$\begin{aligned}\sin\{(-A)+B\} &= \sin\{-(A-B)\} = -\sin(A-B)\\ &= -\sin A\cos B+\cos A\sin B\\ &= \sin(-A)\cos B+\cos(-A)\sin B;\end{aligned}$$

and, in the same way,

$$\sin\{(-A)+(-B)\} = \sin(-A)\cos(-B)+\cos(-A)\sin(-B);$$

the formula is also true generally for *negative* values of the angles involved.

Similar reasoning will, on trial, be found to produce similar results in the case of the three remaining formulæ. Hence the fundamental formulæ all hold for angles generally.

EXERCISE IV.—(1.) Verify the following formulæ by means of those just obtained: $\sin A = \sin(180^\circ - A)$, $\sin\frac{\pi}{2} = 1$.

$$\begin{aligned}\sin(180^\circ - A) &= \sin 180^\circ\cos A - \cos 180^\circ\sin A\\ &= 0\times\cos A-(-1)\sin A = \sin A;\end{aligned}$$

$$\therefore\ \sin A = \sin(180^\circ - A).$$

Also,

$$\begin{aligned}\sin\frac{\pi}{2} &= \sin\left(\frac{\pi}{4}+\frac{\pi}{4}\right)\\ &= \sin\frac{\pi}{4}\cos\frac{\pi}{4}+\cos\frac{\pi}{4}\sin\frac{\pi}{4}\\ &= \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} = \frac{1}{2}+\frac{1}{2};\end{aligned}$$

$$\therefore\ \sin\frac{\pi}{2} = 1;$$

or thus, in this last case,

$$\sin\frac{\pi}{2} = \sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right) = \&c.$$

(2.) Similarly, verify the following: $\sin A = \cos(90^\circ - A)$

$\cos A = \sin(90° \pm A)$, $\cos A = -\cos(180° \pm A)$, $\sin\theta = -\sin(2\pi - \theta)$, $\cos\frac{\pi}{2} = 0$, $\cos 0 = 1$.

(3.) Prove that $\sin 15° = \cos 75° = \frac{\sqrt{3}-1}{2\sqrt{2}}$;

and that $\cos 15° = \sin 75° = \frac{\sqrt{3}+1}{2\sqrt{2}}$.

75. We now proceed to deduce further useful formulæ from those of Arts. 71 and 72. Here and elsewhere, reasoning which is obvious will be frequently omitted in order to save space, and at the same time to afford valuable opportunities for exercise to the student.

76. *Tan $(A \pm B)$ in terms of tan A and tan B.*

Since
$$\frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$
$$= \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A}{\cos A} \cdot \dfrac{\sin B}{\cos B}},$$

where the former numerator and denominator are each divided by $\cos A \cos B$;

therefore
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad \ldots\ldots\ (5).$$

By the same method it appears that
$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \quad \ldots\ldots\ (6).$$

77. *Functions of $2A$ in terms of those of A.*

Since $\sin(A+A) = \sin A \cos A + \cos A \sin A$;

therefore
$$\sin 2A = 2 \sin A \cos A \quad \ldots\ldots\ (7).$$

Similarly it may be shown that
$$\cos 2A = \cos^2 A - \sin^2 A \quad \ldots\ldots\ (8);$$

therefore
$$\cos 2A = 2\cos^2 A - 1 \quad \ldots\ldots\ (9);$$

also
$$\cos 2A = 1 - 2\sin^2 A \quad \ldots\ldots\ (10).$$

Again, since

$$\tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \tan A};$$

therefore $$\tan 2A = \frac{2\tan A}{1-\tan^2 A} \quad \ldots\ldots\ldots\ldots\ldots\ldots(11).$$

EXERCISE V.—(1.) Write out $\cos A$ in terms of $\cos\frac{a}{2}$ and $\sin\frac{a}{2}$, and $\tan 4a$ in terms of $\tan 2a$.

The formula

$$\cos 2a = 2\cos^2 a - 1$$

holds of course for any angles, so long as the angle on one side is double that on the other. Hence

$$\cos a = 2\cos^2\frac{a}{2} - 1.$$

Similarly, since $\tan 2a = \dfrac{2\tan a}{1-\tan^2 a}$,

$$\therefore\ \tan 4a = \frac{2\tan 2a}{1-\tan^2 2a}.$$

(2.) Express $\sin\theta$ in terms of $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$; $\cos\frac{\theta}{2}$ in terms of $\cos\frac{\theta}{4}$ and $\sin\frac{\theta}{4}$; and $\tan\frac{\theta}{3}$ in terms of $\tan\frac{\theta}{6}$.

(3.) Express $\cos 5A$ in terms of $\sin\frac{5A}{2}$, and $\tan 6A$ in terms of $\tan 3A$.

(4.) Given that $\cos\frac{\pi}{2} = 0$, verify that $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

Let $a = \frac{\pi}{4}$, then $2a = \frac{\pi}{2}$, and $\cos 2a = 0$;

therefore, by formula, $1 - 2\sin^2 a = 0$;

transposing, $2\sin^2 a = 1$;

$$\therefore\ \sin^2 a = \frac{1}{2}, \text{ and } \sin a = \frac{1}{\sqrt{2}};$$

$$\therefore\ \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

(5.) Given $\cos 45^\circ = \frac{1}{\sqrt{2}}$, find the value of $\cos 22^\circ 30'$.

(6.) Obtain a numerical value for $\sin 11^\circ 15'$ from that of $\cos 22^\circ 30'$.

(7.) Find expressions for $\cot(A+B)$ and $\cot(A-B)$ in terms of $\cot A$ and $\cot B$, and of $\cot 2A$ in terms of $\cot A$.

(8.) Apply the formula for $\tan(A+B)$ to find the numerical value of $\tan 75°$.

78. *Functions of* $3A$ *in terms of those of* A.

Since

$$\begin{aligned}\sin(2A+A) &= \sin 2A \cos A + \cos 2A \sin A \\ &= 2\sin A \cos^2 A + (1-2\sin^2 A)\sin A \\ &= 2\sin A\,(1-\sin^2 A) + (1-2\sin^2 A)\sin A \\ &= 2\sin A - 2\sin^3 A + \sin A - 2\sin^3 A;\end{aligned}$$

therefore

$$\sin 3A = 3\sin A - 4\sin^3 A.$$

Again, since

$$\begin{aligned}\cos(2A+A) &= \cos 2A \cos A - \sin 2A \sin A \\ &= (2\cos^2 A - 1)\cos A - 2\cos A \sin^2 A \\ &= (2\cos^2 A - 1)\cos A - 2\cos A\,(1-\cos^2 A);\end{aligned}$$

therefore

$$\cos 3A = 4\cos^3 A - 3\cos A.$$

79. *To find sin* $18°$ *and cos* $18°$.

Let $A = 18°$, then $5A = 90°$, and $2A = 90° - 3A$;

therefore $\sin 2A = \sin(90° - 3A) = \cos 3A$;

or $2\sin A \cos A = 4\cos^3 A - 3\cos A$,

dividing both sides by $\cos A$,

$$2\sin A = 4\cos^2 A - 3 = 4 - 4\sin^2 A - 3;$$

transposing, $4\sin^2 A + 2\sin A = 1$;

and extracting root, $\sin A = \dfrac{\pm\sqrt{5}-1}{4}$.

Now $18°$ is in first quadrant, and therefore $\sin 18°$ is positive, therefore

$$\sin 18° = \frac{\sqrt{5}-1}{4};$$

therefore

$$\cos 18° = \sqrt{1-\sin^2 18°} = \frac{\sqrt{10+2\sqrt{5}}}{4} = \cancel{\frac{\sqrt{5}+\sqrt{5}}{2}}.$$

$$= \frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}}$$

80. In order to exhibit certain expedients useful in working examples such as will be given on this Chapter, the following are fully treated.

(1.) Prove that $\tan A + \tan B = \dfrac{\sin(A+B)}{\cos A \cos B}$.

$$\tan A + \tan B = \frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}$$

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B}$$

$$= \frac{\sin(A+B)}{\cos A \cos B}.$$

(2.) Prove the formula

$$\frac{\cos\frac{A}{2} + \sin\frac{A}{2}}{\cos\frac{A}{2} - \sin\frac{A}{2}} = \sec A + \tan A.$$

Multiplying numerator and denominator of the fraction by $\cos\frac{A}{2} + \sin\frac{A}{2}$, we have

$$\frac{\cos\frac{A}{2} + \sin\frac{A}{2}}{\cos\frac{A}{2} - \sin\frac{A}{2}} = \frac{\left(\cos\frac{A}{2} + \sin\frac{A}{2}\right)^2}{\cos^2\frac{A}{2} - \sin^2\frac{A}{2}}$$

$$= \frac{\cos^2\frac{A}{2} + \sin^2\frac{A}{2} + 2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos^2\frac{A}{2} - \sin^2\frac{A}{2}}$$

$$= \frac{1 + \sin A}{\cos A} = \sec A + \tan A.$$

(3.) Show that

$$\sin 2A = 2\cos^2\left(\frac{\pi}{4} - A\right) - 1.$$

$$2\cos^2\left(\frac{\pi}{4} - A\right) - 1 = \cos 2\left(\frac{\pi}{4} - A\right)$$

$$= \cos\left(\frac{\pi}{2} - 2A\right) = \sin 2A.$$

(4.) Expand $\sin(A+B+C)$ in terms of A and B.

$$\begin{aligned}\sin(A+B+C) &= \sin\{(A+B)+C\}\\ &= \sin(A+B)\cos C + \cos(A+B)\sin C\\ &= \sin A\cos B\cos C + \cos A\sin B\cos C\\ &\quad + \cos A\cos B\sin C - \sin A\sin B\sin C.\end{aligned}$$

(5.) Prove that

$$\cos(\theta-\phi) = \frac{\sin(\theta-\alpha)\cos(\phi-\alpha)}{\tan(\theta-\alpha)} + \frac{\cos(\theta-\alpha)\sin(\phi-\alpha)}{\cot(\theta-\alpha)}$$

$$\begin{aligned}&\frac{\sin(\theta-\alpha)\cos(\phi-\alpha)}{\tan(\theta-\alpha)} + \frac{\cos(\theta-\alpha)\sin(\phi-\alpha)}{\cot(\theta-\alpha)}\\ &= \frac{\sin(\theta-\alpha)\cos(\phi-\alpha)}{\dfrac{\sin(\theta-\alpha)}{\cos(\theta-\alpha)}} + \frac{\cos(\theta-\alpha)\sin(\phi-\alpha)}{\dfrac{\cos(\theta-\alpha)}{\sin(\theta-\alpha)}}\\ &= \cos(\theta-\alpha)\cos(\phi-\alpha) + \sin(\theta-\alpha)\sin(\phi-\alpha)\\ &= \cos\{(\theta-\alpha)-(\phi-\alpha)\}\\ &= \cos(\theta-\phi).\end{aligned}$$

EXAMPLES.

Prove the formulæ in 1—34:—

1. $\sin(45^\circ+A) = \dfrac{1}{\sqrt{2}}(\sin A + \cos A)$.

2. $\cos(30^\circ-A) = \dfrac{1}{2}(\sin A + \sqrt{2}\cos A)$.

3. $\operatorname{vers}(A+30^\circ) - \operatorname{vers}(A-30^\circ) = \sin A$.

4. $\dfrac{1-\cos 2A}{\sin 2A} = \tan A$.

5. $\dfrac{1-\cos 2A}{1+\cos 2A} = \tan^2 A$.

6. $\tan(A+45^\circ) = \dfrac{1+\tan A}{1-\tan A}$.

7. $\tan(A-45^\circ) = \dfrac{\tan A-1}{\tan A+1}$.

8. $\tan A - \tan B = \dfrac{\sin(A-B)}{\cos A\cos B}$.

9. $\tan A + \cot A = 2\operatorname{cosec} 2A$.

10. $\tan A - \cot A = -2\cot 2A$.

11. $\cot A - \tan B = \dfrac{\cos(A+B)}{\sin A\cos B}$.

12. $\tan A + \cot B = \dfrac{\cos(A-B)}{\cos A \sin B}$.

13. $2 - 2\tan A \cot 2A = \sec^2 A$.

14. $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$.

15. $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$.

16. $\cos 4A = \dfrac{1}{1+\tan 2A \tan 4A}$.

17. $\cos 2A = \dfrac{\operatorname{cosec}^2 A - 2}{\operatorname{cosec}^2 A}$.

18. $\sin A = 1 - 2\sin^2\left(45^\circ - \dfrac{A}{2}\right)$.

19. $\cos 2A = 2\cos\left(\dfrac{\pi}{4}+A\right)\cos\left(\dfrac{\pi}{4}-A\right)$.

20. $\cos A = \dfrac{\cot\dfrac{A}{2} - \tan\dfrac{A}{2}}{\cot\dfrac{A}{2} + \tan\dfrac{A}{2}}$.

21. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$.

22. $\sin(A+B)\sin(A-B) = \cos^2 B - \cos^2 A$.

23. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$.

24. $\cos(A+B)\cos(A-B) = \cos^2 B - \sin^2 A$.

25. $\dfrac{\cos\left(\theta+\dfrac{\pi}{4}\right)}{\cos\left(\theta-\dfrac{\pi}{4}\right)} = \sec 2\theta - \tan 2\theta$.

26. $\dfrac{1+\sin\alpha}{1-\sin\alpha} = \tan^2\left(\dfrac{\pi}{4}+\dfrac{\alpha}{2}\right)$.

27. $\sin(\theta+2\phi) = \dfrac{\sin\theta}{\sec 2\phi} + \dfrac{\cos\theta}{\operatorname{cosec} 2\phi}$.

28. $\tan(\alpha+30^\circ)\tan(\alpha-30^\circ) = \dfrac{1-2\cos 2\alpha}{1+2\cos 2\alpha}$.

29. $\cos^6\alpha - \sin^6\alpha = \cos 2\alpha\left(1 - \dfrac{1}{4}\sin^2 2\alpha\right)$.

30. $\cos(\alpha+2\beta+3\gamma) = \dfrac{\sin(\alpha+\gamma)\cos 2(\beta+\gamma)}{\tan(\alpha+\gamma)}$
$$- \frac{\cos(\alpha+\gamma)\sin 2(\beta+\gamma)}{\cot(\alpha+\gamma)}.$$

31. $\dfrac{\sin(A+B+C)}{\cos A\cos B\cos C} = \tan A + \tan B + \tan C - \tan A\tan B\tan C.$

32. $\dfrac{\cos(A+B+C)}{\cos A\cos B\cos C} = 1 - \tan B\tan C - \tan C\tan A - \tan A\tan B.$

33. $\dfrac{\sin(A+B+C)}{\sin A\sin B\sin C} = \cot B\cot C + \cot C\cot A + \cot A\cot B - 1.$

34. $\dfrac{\cos(A+B+C)}{\sin A\sin B\sin C} = \cot A\cot B\cot C - \cot A - \cot B - \cot C.$

35. Expand $\tan(A+B+C)$ in terms of $\tan A$, $\tan B$, and $\tan C$.

36. Expand $\cot(A+B+C)$ in terms of $\cot A$, $\cot B$, and $\cot C$.

37. Prove the formula

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A};$$

and employ it to find $\tan 30°$ and $\tan 60°$.

38. Show that

$$\cot(A+B) = \frac{1 - \tan A\tan B}{\tan A + \tan B};$$

and employ it to find $\cot 75°$.

39. Express $\sec(A+B)$ in terms of $\sec A$ and $\sec B$.

40. When $\sin 2A = \cdot 5$, then $\tan\dfrac{A}{2} = 2 - \sqrt{3}$.

41. Given $\sin 18° = \dfrac{\sqrt{5}-1}{4}$, prove that

$$\cos 36° = \sin 54° = \frac{\sqrt{5}+1}{4}.$$

42. Given $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$, find $\tan\dfrac{\pi}{2}$.

43. Deduce the expression for $\sin(A-B)$ from that for $\sin(A+B)$ in terms of A and B.

44. Given $\cos(A-B) = \cos A\cos B + \sin A\sin B$, deduce the expression for $\sin(A-B)$, and also that

$$\sin^2 A + \cos^2 A = 1.$$

45. If $A+B$ is $< \dfrac{\pi}{2}$, then $\cot A\cot B$ is < 1.

CHAPTER X.

FORMULÆ FOR TRANSFORMATIONS OF SUMS, DIFFERENCES, AND PRODUCTS OF FUNCTIONS.—INVERSE FUNCTIONS.

81. Certain formulæ will now be treated of, which are of great service in simplifying Trigonometrical transformations. They will possibly, on first reading, be found complex; but it is indispensably necessary that they should be thoroughly understood and known familiarly.

82. Because

$$\sin A \cos B + \cos A \sin B = \sin(A+B),$$

and $$\sin A \cos B - \cos A \sin B = \sin(A-B);$$

therefore by addition,

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B) \quad \ldots\ldots(1);$$

and by subtraction,

$$2 \cos A \sin B = \sin(A+B) - \sin(A-B) \quad \ldots\ldots(2).$$

Again, because

$$\cos A \cos B - \sin A \sin B = \cos(A+B),$$

and $$\cos A \cos B + \sin A \sin B = \cos(A-B);$$

therefore by addition,

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B) \quad \ldots\ldots(3);$$

and by subtraction,

$$\left.\begin{aligned} -2 \sin A \sin B &= \cos(A+B) - \cos(A-B) \\ \text{or}\quad 2 \sin A \sin B &= \cos(A-B) - \cos(A+B) \end{aligned}\right\} \quad \ldots\ldots(4).$$

It will be found well worth while to learn these

formulæ as given above, and also expressed verbally as follows:—

Of any two angles,

$$2 \sin \textit{one} \,.\, \cos \textit{the other} = \sin \textit{sum} + \sin \textit{difference} \ldots (1).$$

$$2 \cos \textit{one} \,.\, \sin \textit{the other} = \sin \textit{sum} - \sin \textit{difference} \ldots (2).$$

$$2 \cos \textit{one} \,.\, \cos \textit{the other} = \cos \textit{sum} + \cos \textit{difference} \ldots (3).$$

$$\left.\begin{aligned} -2 \sin \textit{one} \,.\, \sin \textit{the other} &= \cos \textit{sum} - \cos \textit{difference} \\ \text{or } 2 \sin \textit{one} \,.\, \sin \textit{the other} &= \cos \textit{difference} - \cos \textit{sum} \end{aligned}\right\} (4).$$

83. As before seen,

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B.$$

Since here A and B may have any values, put $\frac{A+B}{2}$ for A and $\frac{A-B}{2}$ for B. We shall then require to replace $A+B$ by A, for

$$\frac{A+B}{2} + \frac{A-B}{2} = A;$$

and $A-B$ by B, for

$$\frac{A+B}{2} - \frac{A-B}{2} = B.$$

Hence, $\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} \ldots (5).$

Applying the same process to (2), (3), (4),

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \ldots\ldots\ldots (6).$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} \ldots\ldots\ldots (7).$$

$$\left.\begin{aligned} \cos A - \cos B &= -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2} \\ \text{or} \quad \cos B - \cos A &= 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2} \end{aligned}\right\} \ldots (8).$$

A and B are any angles; and we may therefore say:

Of any two angles,

sin *one* + sin *the other*

$$= 2 \sin \textit{semi-sum} \,.\, \cos \textit{semi-difference} \ldots (5).$$

sin *one* − sin *the other*

$$= 2 \cos \textit{semi-sum} \,.\, \sin \textit{semi-difference} \ldots (6).$$

cos *one* + cos *the other*

$$= 2 \cos \textit{semi-sum} \,.\, \cos \textit{semi-difference} \ldots (7).$$

cos *one* − cos *the other*

$$= -2 \sin \textit{semi-sum} \,.\, \sin \textit{semi-difference} \ldots (8).$$

EXERCISE VI.—(1.) Given formulæ (1) and (2) above, verify that

$$\sin A \cos B + \cos A \sin B = \sin (A+B).$$

$$\sin A \cos B + \cos A \sin B$$

$$= \frac{2 \sin A \cos B + 2 \cos A \sin B}{2}$$

$$= \frac{\sin (A+B) + \sin (A-B) + \sin (A+B) - \sin (A-B)}{2}$$

$$= \frac{2 \sin (A+B)}{2} = \sin (A+B).$$

(2.) Similarly verify formulæ (2), (3), (4), (7) of Chap. IX., given (1), (2), (3), (4) of Chap. X.

(3.) Show fully that

$$2 \sin (A+B) \cos (A-B) = \sin 2A + \sin 2B.$$

Here, "sum" $= A+B+A-B = 2A$;

and "difference" $= A+B-(A-B)$

$= A+B-A+B = 2B$;

$$\therefore\ 2 \sin (A+B) \cos (A-B) = \sin 2A + \sin 2B.$$

(4.) Show fully that

(i.) $2 \cos (\theta+\phi) \sin (\theta-\phi) = \sin 2\theta - \sin 2\phi.$

(ii.) $2 \cos (2\alpha+\beta) \cos (\alpha+2\beta) = \cos 3 (\alpha+\beta) + \cos (\alpha-\beta).$

(iii.) $2 \sin 4A \sin 3A = \cos A - \cos 7A.$

(iv.) $2 \sin \left(\frac{\pi}{4} + A\right) \cos \left(\frac{\pi}{4} - A\right) = 1 + \sin 2A.$

(v.) $2\cos\left(\frac{\pi}{2}+\frac{A}{2}\right)\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)=\cos A-1.$

(vi.) $4\sin(60^\circ-A)\sin(60^\circ+A)=2\cos 2A+1.$

(5.) Apply formula (5) to prove that

$$\sin(30^\circ+A)+\sin(30^\circ-A)=\cos A.$$

Here, "semi-sum" $=\frac{1}{2}\left\{30^\circ+A+30^\circ-A\right\}=30^\circ,$

and "semi-difference" $=\frac{1}{2}\left\{30^\circ+A-(30^\circ-A)\right\}$

$$=\frac{1}{2}\left\{30^\circ+A-30^\circ+A\right\}=A.$$

$$\therefore \sin(30^\circ+A)+\sin(30^\circ-A)=2\sin 30^\circ\cos A$$
$$=2\times\frac{1}{2}\cos A$$
$$=\cos A.$$

(6.) Prove in a similar manner,

(i.) $\cos(30^\circ-A)-\cos(30^\circ+A)=\sin A.$

(ii.) $\sin(60^\circ+A)+\sin(60^\circ-A)=\sqrt{3}\cos A.$

(iii.) $\sin\left(\frac{\pi}{4}+\theta\right)-\sin\left(\frac{\pi}{4}-\theta\right)=\sqrt{2}\sin\theta.$

(iv.) $\sin 2A+\sin 4B=2\sin(A+2B)\cos(A-2B).$

(v.) $\cos(2\alpha-3\beta)+\cos 5\beta=2\cos(\alpha+\theta)\cos(\alpha-4\beta).$

(vi.) $\cos\frac{\theta}{3}-\cos\frac{\theta}{2}=2\sin\frac{5\theta}{12}\sin\frac{\theta}{12}.$

84. *Sin* $\frac{A}{2}$ *and cos* $\frac{A}{2}$ *in terms of sin* A.

Because $\sin^2\frac{A}{2}+\cos^2\frac{A}{2}=1,$

and $2\sin\frac{A}{2}\cos\frac{A}{2}=\sin A;$

adding, $\sin^2\frac{A}{2}+\cos^2\frac{A}{2}+2\sin\frac{A}{2}\cos\frac{A}{2}=1+\sin A;$

subtracting, $\sin^2 \frac{A}{2} + \cos^2 \frac{A}{2} - 2 \sin \frac{A}{2} \cos \frac{A}{2} = 1 - \sin A$;

extracting roots, $\sin \frac{A}{2} + \cos \frac{A}{2} = \pm \sqrt{1 + \sin A}$ ..(9),

and $\sin \frac{A}{2} - \cos \frac{A}{2} = \pm \sqrt{1 - \sin A}$...(10);

adding (10) to (9), and subtracting (10) from (9), we obtain

$$2 \sin \frac{A}{2} = \pm \sqrt{1 + \sin A} \pm \sqrt{1 - \sin A} \quad ...(11);$$

and $$2 \cos \frac{A}{2} = \pm \sqrt{1 + \sin A} \mp \sqrt{1 - \sin A} \quad ...(12).$$

When the value of A, or the quadrant to which it belongs, is given, formulæ (11) and (12) may be obtained with single instead of double signs. This is done by first determining the proper signs for the radicals of (9) and (10), and then adding and subtracting. The mode of determining the proper signs for the radicals of (9) and (10) is fully shown in Art. 60, which should be re-read along with this article. Also see Ex. 8. Art. 86.

85. Inverse Functions.—If $\sin \theta = a$, then θ may be described as *the angle of which the sine is a.* In order to express briefly an angle when it is required to speak of it in this manner, it is usual to write

$$\sin^{-1} a$$

for the words "*the angle whose sine is a.*" And similarly if $\tan a = t$, $\sec \beta = s$, then

$$a = \tan^{-1} t, \quad \beta = \sec^{-1} s;$$

where $\tan^{-1} t$, $\sec^{-1} s$ would be respectively read, "the angle whose tangent is t," and "the angle whose secant is s."

$\text{Sin}^{-1} a$, $\cos^{-1} b$, $\tan^{-1} c$, &c., must be distinguished from $(\sin a)^{-1}$, $(\cos b)^{-1}$, $(\tan c)^{-1}$, &c.; that is, from $\frac{1}{\sin a}$, $\frac{1}{\cos b}$, $\frac{1}{\tan c}$, &c. In Trigonometry we write

$\sin^2 a$, $\cos^3 \beta$, and so on, where 2, 3, are indices; but -1 is *not an index* in the above expressions, $\sin^{-1} a$, $\cos^{-1} b$, &c., but has the meaning already explained.

86. The following are specimens of working out such as will be required in the examples which follow them.

(1.) Prove that

$$3 \sin A - \sin 3A = 2 \sin A\,(1 - \cos 2A).$$

$$\begin{aligned} 3 \sin A - \sin 3A &= 2 \sin A - (\sin 3A - \sin A) \\ &= 2 \sin A - 2 \cos 2A \sin A \\ &= 2 \sin A\,(1 - \cos 2A). \end{aligned}$$

(2.) Prove that

$$\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta.$$

$$\begin{aligned} \sin^2\alpha - \sin^2\beta &= (\sin\alpha + \sin\beta)(\sin\alpha - \sin\beta) \\ &= 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \cdot 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\ &= 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2} \cdot 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha-\beta}{2} \\ &= \sin(\alpha+\beta)\sin(\alpha-\beta). \end{aligned}$$

(3.) Prove that $\dfrac{\sin\theta + \sin 2\theta + \sin 3\theta}{\cos\theta + \cos 2\theta + \cos 3\theta} = \tan 2\theta.$

$$\begin{aligned} \frac{\sin\theta + \sin 2\theta + \sin 3\theta}{\cos\theta + \cos 2\theta + \cos 3\theta} &= \frac{\sin\theta + \sin 3\theta + \sin 2\theta}{\cos\theta + \cos 3\theta + \cos 2\theta} \\ &= \frac{2\sin 2\theta\cos\theta + \sin 2\theta}{2\cos 2\theta\cos\theta + \cos 2\theta} \\ &= \frac{\sin 2\theta\,(2\cos\theta + 1)}{\cos 2\theta\,(2\cos\theta + 1)} \\ &= \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta. \end{aligned}$$

(4.) Prove that $\sin(A-B)\sin(C-D) + \sin(A-D)\sin(B-C) + \sin(A-C)\sin(D-B) = 0.$

$$\begin{aligned} &\sin(A-B)\sin(C-D) + \sin(A-D)\sin(B-C) + \\ &\qquad \sin(A-C)\sin(D-B) \\ &= \frac{2\sin(A-B)\sin(C-D) + 2\sin(A-D)\sin(B-C)}{2} \\ &\qquad + \frac{2\sin(A-C)\sin(D-B)}{2} \end{aligned}$$

$$= \frac{\cos(A-B-C+D) - \cos(A-B+C-D) + \cos(A-B+C-D)}{2}$$

$$\frac{-\cos(A+B-C-D) + \cos(A+B-C-D) - \cos(A-B-C+D)}{2}$$

$$= \frac{0}{2} = 0.$$

(5.) If $A+B+C = 180°$, then

$$\sin A + \sin B + \sin C = 4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}.$$

Here $C = 180° - (A+B)$, $\therefore \sin C = \sin(A+B)$;

and $\frac{C}{2} = 90° - \frac{A+B}{2}$, $\therefore \cos\frac{C}{2} = \sin\frac{A+B}{2}$;

$$\therefore \sin A + \sin B + \sin C = \sin A + \sin B + \sin(A+B)$$

$$= 2 \sin\frac{A+B}{2} \cos\frac{A-B}{2} + 2 \sin\frac{A+B}{2} \cos\frac{A+B}{2}$$

$$= 2 \sin\frac{A+B}{2}\left(\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right)$$

$$= 2 \cos\frac{C}{2} \times 2 \cos\frac{A}{2} \cos\frac{B}{2}$$

$$= 4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}.$$

(6.) If $A+B+C = 90°$, then

$$\cot A + \cot B + \cot C = \cot A \cot B \cot C.$$

It appeared by Chap. IX., that

$$\cot(A+B+C) = \frac{\cot A \cot B \cot C - \cot A - \cot B - \cot C}{\cot A \cot B + \cot A \cot C + \cot B \cot C - 1}.$$

But $\cot(A+B+C) = \cot 90° = 0$;

$$\therefore \cot A \cot B \cot C - \cot A - \cot B - \cot C = 0;$$

$$\therefore \cot A + \cot B + \cot C = \cot A \cot B \cot C.$$

(7.) Solve the equation $\cos 5\theta - \cos\theta = \sin 3\theta$.

Since $\cos 5\theta - \cos\theta = \sin 3\theta$,

or, $-2 \sin 3\theta \sin 2\theta = \sin 3\theta$;

therefore (i.) $\sin 3\theta = 0 = \sin n\pi$,

$$\therefore 3\theta = n\pi;$$

and (ii.)
$$-2\sin 2\theta = 1;$$
$$\therefore\ \sin 2\theta = -\frac{1}{2} = \sin\frac{7\pi}{6}$$
$$= \sin\left\{n\pi + (-1)^n\frac{7\pi}{6}\right\}.$$
$$\therefore\ 2\theta = n\pi + (-1)^n.\ \frac{7\pi}{6}.$$

Hence
$$\theta = \frac{n\pi}{3},\ \text{or}\ \frac{n\pi}{2} + (-1)^n\frac{7\pi}{12}$$

(8.) Given $\sin 330^\circ = -\frac{1}{2}$, find $\sin 165^\circ$ and $\cos 165^\circ$.

Referring to formulæ (9) and (10), and noting that $\cos 165^\circ$ is greater than $\sin 165^\circ$ and negative:
$$\sin 165^\circ + \cos 165^\circ = -\sqrt{1+\sin 330^\circ} = -\sqrt{\frac{1}{2}},$$
and
$$\sin 165^\circ - \cos 165^\circ = +\sqrt{1-\sin 330^\circ} = +\sqrt{\frac{3}{2}}.$$
Adding and subtracting,
$$2\sin 165^\circ = \frac{\sqrt{3}-1}{\sqrt{2}},\qquad \therefore\ \sin 165^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}};$$
and
$$2\cos 165^\circ = \frac{-\sqrt{3}-1}{\sqrt{2}},\quad \therefore\ \cos 165^\circ = -\frac{\sqrt{+31}}{2\sqrt{2}}.$$

(9.) If $\sin B = m\sin(2A+B)$, then
$$\tan(A+B) = \frac{1+m}{1-m}\tan A.$$

Because
$$\sin B = m\sin(2A+B),$$
$$\therefore\ \frac{1}{m} = \frac{\sin(2A+B)}{\sin B};$$
then by theory of proportion ("componendo et dividendo"),
$$\frac{1+m}{1-m} = \frac{\sin(2A+B)+\sin B}{\sin(2A+B)-\sin B}$$
$$= \frac{2\sin(A+B)\cos A}{2\cos(A+B)\sin A} = \frac{\tan(A+B)}{\tan A};$$
$$\therefore\ \frac{1+m}{1-m}\tan A = \tan(A+B).$$

(10.) Show that $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5} = \frac{\pi}{2}$.

$$\text{Let} \quad \alpha = \sin^{-1}\frac{3}{5}, \quad \text{or} \quad \sin\alpha = \frac{3}{5},$$

$$\text{and} \quad \beta = \sin^{-1}\frac{4}{5}, \quad \text{or} \quad \sin\beta = \frac{4}{5};$$

$$\therefore \quad \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \frac{3}{5}\sqrt{\left\{1-\left(\frac{4}{5}\right)^2\right\}} + \frac{4}{5}\sqrt{\left\{1-\left(\frac{3}{5}\right)^2\right\}}$$

$$= \frac{3}{5}\times\frac{3}{5} + \frac{4}{5}\times\frac{4}{5}$$

$$= \frac{9}{25} + \frac{16}{25} = 1 = \sin\frac{\pi}{2}.$$

$$\therefore \quad \alpha+\beta = \frac{\pi}{2},$$

or $$\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5} = \frac{\pi}{2}.$$

(11.) Prove that $2\tan^{-1}\frac{1}{2} + \cos^{-1}\frac{4}{5} = \frac{\pi}{2}$.

$$2\tan^{-1}\frac{1}{2} = \tan^{-1}\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} = \tan^{-1}\frac{4}{3};$$

and $$\cos^{-1}\frac{4}{5} = \sec^{-1}\frac{5}{4} = \tan^{-1}\sqrt{\left(\frac{25}{16}-1\right)} = \tan^{-1}\frac{3}{4};$$

$$\therefore \quad 2\tan^{-1}\frac{1}{2} + \cos^{-1}\frac{4}{5} = \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{3}{4}$$

$$= \tan^{-1}\frac{\frac{4}{3}+\frac{3}{4}}{1-\frac{4}{3}\times\frac{3}{4}} = \tan^{-1}\frac{\frac{4}{3}+\frac{3}{4}}{1-1}$$

$$= \tan^{-1}\infty = \frac{\pi}{2}.$$

(12.) Solve the equation

$$\sin^{-1}2x = \sin^{-1}x\sqrt{3} + \sin^{-1}x.$$

Since $$\sin^{-1}2x = \sin^{-1}x\sqrt{3} + \sin^{-1}x$$

$$= \sin^{-1}(x\sqrt{3}\,.\sqrt{1-x^2} + x\sqrt{1-3x^2});$$

$$\therefore \quad 2x = x\sqrt{3(1-x^2)} + x\sqrt{1-3x^2}.$$

Therefore (i.) $x = 0$.

(ii.) $2 = \sqrt{3(1-x^2)} + \sqrt{1-3x^2}$;

solving which equation, $x = \pm \frac{1}{2}$.

Hence required roots are 0 and $\pm \frac{1}{2}$.

EXAMPLES.

Formulæ 1—8 of this chapter may be applied with advantage in all the following examples from 1—45.

Prove the formulæ in 1—39.

1. $\cos(60° + A) + \cos(60° - A) = \cos A$.

2. $\sin A + \sin 2A = 2 \sin \frac{3A}{2} \sin\left(\frac{\pi}{2} - \frac{A}{2}\right)$.

3. $4 \cos(30° + A) \cos(30° - A) = 2 \cos 2A + 1$.

4. $2 \sin A \cos 2B = \sin(A + 2B) - \sin(2B - A)$.

5. $\sin 2A - \sin 3A + \sin 4A = \sin 3A\,(2 \cos A - 1)$.

6. $\cos\theta + \cos\frac{3\theta}{4} + \cos\frac{\theta}{2} = \cos\frac{3\theta}{4}\left(2\cos\frac{\theta}{4} + 1\right)$.

7. $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan \frac{1}{2}(A + B)$.

8. $\dfrac{\sin A - \sin B}{\cos A + \cos B} = \tan \frac{1}{2}(A - B)$.

9. $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \tan\frac{A+B}{2} \cot\frac{A-B}{2}$.

10. $\dfrac{\cos B - \cos A}{\cos B + \cos A} = \tan\frac{A+B}{2} \tan\frac{A-B}{2}$.

11. $\dfrac{\cos A - \cos 3A}{\sin 3A - \sin A} = \tan 2A$.

12. $\dfrac{\sin(A+B) + \sin(A-B)}{\cos(A+B) - \cos(A-B)} = -\cot B$.

13. $\dfrac{\sin\theta + \sin 2\theta}{\cos\theta - \cos 2\theta} = \cot\frac{\theta}{2}$.

14. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$.

15. $\cos\beta \cos(\alpha + 2\beta) = \cos^2(\alpha + \beta) - \sin^2\alpha$.

16. $\dfrac{\sin(\alpha+\beta)\sin(\alpha-\beta)}{\cos^2\alpha \cos^2\beta} = \tan^2\alpha - \tan^2\beta$.

17. $\dfrac{\cos 2a - \cos 4a}{\sin 4a - \sin 2a} = \tan 3a.$

18. $\dfrac{\sin 2A + \sin A}{\cos 2A + \cos A} = \tan \dfrac{3A}{2}.$

19. $\dfrac{\sin a + 2\sin 3a + \sin 5a}{\sin 3a + 2\sin 5a + \sin 7a} = \dfrac{\sin 3a}{\sin 5a}.$

20. $\tan 2\theta - \tan \theta = \dfrac{2\sin \theta}{\cos \theta + \cos 3\theta}.$

21. $\tan 2A + \tan A = \dfrac{2\sin 3A}{\cos A + \cos 3A}.$

22. $\cos (A+B) \sin B - \cos (A+C) \sin C$
$= \sin (A+B) \cos B - \sin (A+C) \cos C.$

23. $\cos (A+B) \sin (A-B) + \cos (B+C) \sin (B-C)$
$+ \cos (C+D) \sin (C-D) + \cos (D+A) \sin (D-A) = 0.$

24. $\sin (A+B-C) + \sin (A+C-B) + \sin (B+C-A)$
$- \sin (A+B+C) = 4\sin A \sin B \sin C.$

25. $\cos (A+B-C) + \cos (A+C-B) + \cos (B+C-A)$
$+ \cos (A+B+C) = 4\cos A \cos B \cos C.$

26. $4\cos A \cos (120^\circ - A) \cos (120^\circ + A) = \cos 3A.$

27. $\cos 10A + \cos 8A + 3\cos 4A + 3\cos 3A = 8\cos A \cos^3 2A.$

28. $\cos \theta + \cos \left(\dfrac{2\pi}{3} - \theta\right) + \cos \left(\dfrac{2\pi}{3} + \theta\right) = 0.$

29. $4\sin \theta \sin \left(\dfrac{\pi}{3} - \theta\right) \sin \left(\dfrac{\pi}{3} + \theta\right) = \sin 3\theta.$

30. $4\cos^3 \theta \sin 3\theta + 4\sin^3 \theta \cos 3\theta = 3\sin 4\theta.$

31. $\sin 3\theta \sin^3 \theta + \cos 3\theta \cos^3 \theta = \cos^3 2\theta.$

In 32—37, $A+B+C = 180^\circ$.

32. $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C.$

33. $\cos A + \cos B + \cos C = 1 + 4\sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}.$

34. $\sin A - \sin B + \sin C = 4\sin \dfrac{A}{2} \cos \dfrac{B}{2} \sin \dfrac{C}{2}.$

35. $\cos \dfrac{A}{2} + \cos \dfrac{B}{2} + \cos \dfrac{C}{2} = 4\cos \dfrac{\pi - A}{4} \cos \dfrac{\pi - B}{4} \cos \dfrac{\pi - C}{4}.$

36. $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1.$

37. $\dfrac{\sin A + \sin B - \sin C}{\sin A + \sin B + \sin C} = \tan\dfrac{A}{2}\tan\dfrac{B}{2}$.

In 38, 39, $A+B+C=90°$.

38. $\tan A \tan B + \tan A \tan C + \tan B \tan C = 1$.

39. $\tan A + \tan B + \tan C = \tan A \tan B \tan C + \sec A \sec B \sec C$.

Solve the equations in 40—53.

40. $\sin 2\theta + \sin\theta = 0$.

41. $\sin 3\theta + \sin 2\theta + \sin\theta = 0$.

42. $\cos 3\theta + \cos 2\theta + \cos\theta = 0$.

43. $\cos\theta + \cos\dfrac{\theta}{2} = \cos\dfrac{3\theta}{4}$.

44. $\sin 7\theta - \sin\theta = \sin 3\theta$.

45. $\cos n\theta + \cos(n-2)\theta = \cos\theta$.

46. $\sin\theta + \cos\theta = \dfrac{1}{\sqrt{2}}$.

47. $\sin\theta + \cos\theta = \sqrt{2}$.

48. $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$.

49. $\tan a + \tan\left(\dfrac{\pi}{4} + a\right) = 2$.

50. $\tan\left(\dfrac{\pi}{4} + a\right)\cot\left(\dfrac{\pi}{4} - a\right) = 3$.

51. $\cos a \cos 3a = \cos 5a \cos 7a$.

52. $\sin a - \cos a = 4\sin a \cos^2 a$.

53. $\sec a - \operatorname{cosec} a = 2\sqrt{2}$.

54. Given $\sin 18° = \dfrac{\sqrt{5}-1}{4}$, find $\sin 9°$ and $\cos 9°$.

55. Given $\sin 210° = -\dfrac{1}{2}$, find $\sin 105°$ and $\cos 105°$.

56. Given $\sin 15° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$, find $\sin 7° 30'$ and $\cos 7° 30'$.

57. Prove that

$$\operatorname{versin} a \operatorname{versin}(\beta+\gamma) = \sin a \sin(\beta+\gamma),$$

when $a+\beta+\gamma=\pi$.

58. If A, B, C be angles of a triangle, show that

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C;$$

and employ the result to find $\tan 60°$.

59. If $a \cos \beta = b \cos \alpha$, prove that

$$\frac{a+b}{a-b} = \cot \frac{\alpha+\beta}{2} \cot \frac{\alpha-\beta}{2}.$$

60. If $2 \sec 2\alpha = \tan \beta + \cot \beta$, then

$$\alpha + \beta = \frac{\pi}{4}.$$

61. If $\sec (\theta + \alpha) + \sec (\theta - \alpha) = 2 \sec \theta$, show that

$$\cos \theta = \sqrt{2} \cos \frac{\ }{2}.$$

62. If $\tan \theta = \frac{b}{a}$, then $a \cos 2\theta + b \sin 2\theta = a$.

63. If the sum of two angles be given, show that the sum of their sines is greatest, and the sum of their tangents is least, when the angles are equal.

64. Prove that

$$\frac{2 \cos 2A}{\cos A + \sin A} + \frac{2 \sin 2A}{\cos A - \sin A} = \frac{\sqrt{2}}{\cos\left(A + \frac{\pi}{4}\right)}.$$

Prove as formulæ 65—81.

65. $\cos^{-1} x = \sec^{-1} \frac{1}{x}$.

66. $\sin^{-1} x = \cos^{-1} \sqrt{1-x^2}$.

67. $\cot^{-1} x = \operatorname{cosec}^{-1} \sqrt{1+x^2}$.

68. $\cos^{-1} x = 2 \cos^{-1} \sqrt{\left(\frac{1+x}{2}\right)}$.

69. $\tan^{-1} t_1 + \tan^{-1} t_2 = \tan^{-1} \frac{t_1 + t_2}{1 - t_1 t_2}$.

70. $\tan^{-1} \frac{3}{5} + \cot^{-1} \frac{7}{3} = \cot^{-1} \frac{13}{18}$.

71. $\sin^{-1} \frac{1}{2} + \sin^{-1} \frac{1}{\sqrt{2}} = 75°$.

72. $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.

73. $\tan \left(\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}\right) = 1$.

74. $2 \tan^{-1} \frac{1}{3} = \cot^{-1} \frac{4}{3}$.

75. $\tan^{-1} \frac{2}{3} = \frac{1}{2} \tan^{-1} \frac{12}{5}$.

76. $\cot\left(\cot^{-1}\frac{3}{4} + \cot^{-1}\frac{1}{7}\right) = -1.$

77. $\cos^{-1}\frac{1}{2} + 2\sin^{-1}\frac{1}{2} = \frac{2\pi}{3}.$

78. $\tan^{-1}\frac{2x-y}{y\sqrt{3}} + \tan^{-1}\frac{2y-x}{x\sqrt{3}} = \frac{\pi}{3}.$

79. $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{\pi}{2}.$

80. $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}.$

81. $\tan(2\tan^{-1}x) = 2\tan(\tan^{-1}x + \tan^{-1}x^3).$

Solve 82—85 as equations.

82. $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}.$

83. $\tan^{-1}x + \cot^{-1}(-x) = \pi.$

84. $\sin^{-1}\frac{2a}{1+a^2} + \sin^{-1}\frac{2b}{1+b^2} = 2\tan^{-}x.$

85. $\cot^{-1}\frac{1}{x+1} + \cot^{-1}\frac{1}{x} + \cot^{-1}\frac{1}{x-1} = \tan^{-1}3x.$

86. If $\sec^{-1}a + \sec^{-1}\frac{x}{a} = \sec^{-1}b + \sec^{-1}\frac{x}{b}$, then $x = \pm ab$.

87. If $\operatorname{versin}^{-1}(1+x) - \operatorname{versin}^{-1}(1-x) = \tan^{-1}2\sqrt{1-x^2}$, then

$$x = \pm 1, \text{ or } -\frac{1}{2}.$$

88. Distinguish between

$$\sin^{-1}\tan\frac{\pi}{4} \text{ and } \tan\sin^{-1}\frac{1}{\sqrt{2}}$$

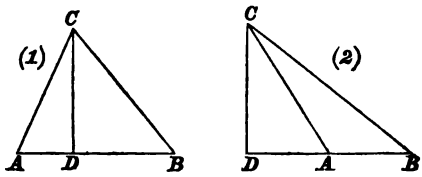
CHAPTER XI.

SOLUTION OF TRIANGLES WITHOUT LOGARITHMS.

87. THE useful properties of right-angled triangles have already been investigated. We shall now turn our attention to those of triangles generally. As before stated, in such investigations we are accustomed to write A, B, C for the angles of a triangle, and a, b, c for the sides respectively opposite to them.

88. *In any triangle the sides are respectively proportional to the sines of the opposite angles.*

Let ABC be any triangle. Draw CD perpendicular to AB, falling (1) within the triangle, and (2) without it.



In (2) the angles CAB and CAD are supplements, therefore

$$\sin CAB = \sin CAD.$$

Hence, in (2) as well as in (1),

$$\frac{CD}{CA} = \sin A, \quad \therefore\ CD = CA \sin A = b \sin A;$$

and $$\frac{CD}{CB} = \sin B, \quad \therefore\ CD = CB \sin B = a \sin B;$$

therefore $$b \sin A = a \sin B;$$

therefore $$\frac{\sin A}{\sin B} = \frac{a}{b}, \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b}.$$

In the same way we may show that

$$\frac{\sin A}{\sin C} = \frac{a}{c}, \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin C}{c}.$$

Wherefore it follows that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}:$$

which may also be written

$$\sin A : \sin B : \sin C = a : b : c.$$

89. *To express a side of a triangle in terms of the other sides and the adjacent angles.*

This may form a corollary to the foregoing.

Since in ABC, by Euc. I. 32,

$$A+B+C = 180°, \quad \therefore A = 180° - (B+C);$$

and therefore $\sin A = \sin (B+C)$

$$= \sin B \cos C + \cos B \sin C;$$

therefore $1 = \frac{\sin B}{\sin A} \cos C + \frac{\sin C}{\sin A} \cos B$

$$= \frac{b}{a} \cos C + \frac{c}{a} \cos B;$$

wherefore $a = b \cos C + c \cos B.$

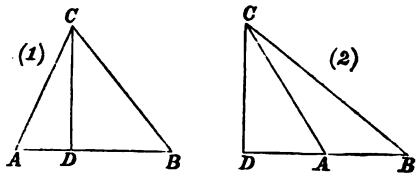
We might thus prove also that

$$b = a \cos C + c \cos A,$$

and $c = a \cos B + b \cos A.$

90. *To express any side of a triangle in terms of the other two sides and the angle between them.*

Constructing as in Art. 88, in (1) by Euc. II. 13,



$$CB^2 = AC^2 + AB^2 - 2AB \cdot AD;$$

but $$AD = AC\frac{AD}{AC} = AC \cos A;$$

therefore $$CB^2 = AC^2 + AB^2 - 2AB \,.\, AC \cos A.$$

In (2), by Euc. II. 12,

$$CB^2 = AC^2 + AB^2 + 2AB \,.\, AD;$$

but $$AD = AC\frac{AD}{AC} = AC \cos(180° - A) = -AC \cos A;$$

therefore

$$CB^2 = AC^2 + AB^2 - 2AB \,.\, AC \cos A.$$

Wherefore in every triangle

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Similarly $$b^2 = a^2 + c^2 - 2ac \cos B,$$

and $$c^2 = a^2 + b^2 - 2ab \cos C.$$

91. *To express the cosine of an angle of a triangle in terms of the three sides.*

Transposing in the formula

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

we have $$2bc \cos A = b^2 + c^2 - a^2;$$

therefore $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Similarly $$\cos B = \frac{a^2 + c^2 - b^2}{2ac},$$

and $$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

92. It will now be possible to enter upon the solution of oblique-angled triangles. In general, it would require formulæ which we have yet to obtain in order to solve such triangles without great expenditure of labour. Nevertheless we may undertake examples which do not involve large or complex quantities; and we shall proceed to illustrate the nature of this part of our subject by doing so.

93. The formulæ we shall employ are

$$A + B + C = 180° \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{.....(2)},$$

$$a^2 = b^2 + c^2 - 2bc \cos A \quad \text{.....(3)},$$

and $$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \text{.....(4)}.$$

It will now be shown that they are sufficient.

Three parts of a triangle are always given from which to find the others; and the three parts can never be the three angles. All the cases of solution are these :—

(i.) *Given two angles and a side.*

Let A, B, a be the given parts of the triangle.

Angle C may be found by applying (1), by which

$$C = 180° - (A + B),$$

and sides b and c will then both follow from (2), by which

$$b = a \frac{\sin B}{\sin A}, \qquad c = a \frac{\sin C}{\sin A}.$$

In obtaining c, we had a choice of modes of proceeding, as very often happens. We were at liberty to apply (3) instead of (2), but obviously to less advantage.

(ii.) *Given two sides and the angle opposite to one of them.*

Let a, b, A be given.

Angle B may be found by applying (2), (but see Art. 94), by which

$$\sin B = \frac{b}{a} \sin A;$$

C is then given by (1), and c will follow from (2), by which

$$c = b \frac{\sin C}{\sin B}.$$

(iii.) *Given two sides and the angle between them.*

Let a, b, C be given.

Side c may be found by applying (3), by which

$$c^2 = a^2 + b^2 - 2ab \cos C;$$

A is then given by (2) or (4), and B afterwards by (1).

(iv.) *Given the three sides.*

The parts are a, b, c; therefore

A may be formed by (4); then B by (2) or (4); and afterwards C by (1).

94. *The Ambiguous Case.* Case (ii.) will require further attention, and will be fully stated. Under certain conditions it is called the *Ambiguous Case.*

Given two sides and the angle opposite to one of them, to solve the triangle.

Let a, b, A be the parts given; then B, C, c are those required.

Without first knowing B, we cannot find C, c. To find B we have

$$\sin B = \frac{b}{a} \sin A.$$

If $\frac{b}{a} \sin A = 1 = \sin 90°$, then $B = 90°$. Otherwise $\frac{b}{a} \sin A$, whatever be its value, is the sine of two angles which are supplements to each other, one acute, the other obtuse. Which of these must be taken as the proper magnitude of B can only be known from the given values of a and b, *if at all.* It is

(1) *Unknown* when $a < b$, or the side opposite to the required angle is the greater of the two given sides.

For then $B > A$, and hence may, for anything we know, be either acute or obtuse.

In this case, therefore, two magnitudes of B are possible.

Since these give two values each for C, c, we infer that there are two triangles, which have three of their parts equal to a, b, A respectively; that is, which satisfy the conditions of the problem.

Wherefore in this case the solution is *ambiguous.*

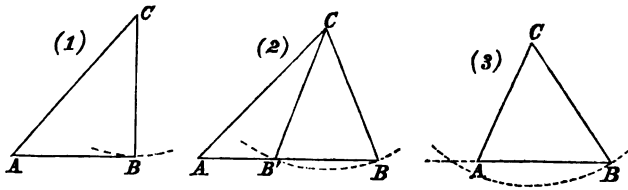
(2) *Known* when $a > b$, or the side opposite to the given angle is the greater of the two given sides.

For then by Euc. I. 18, $A > B$;
therefore by Euc. I. 17, $B < 90°$;
and there is therefore no ambiguity.

95. *The ambiguous case geometrically illustrated.*

It will be profitable to consider in connection with the foregoing the geometrical ambiguity which corresponds to it.

Given A, a, b, *to* construct *the triangle.*



Make the straight line CA equal to b. At point A in CA make angle CAB equal to A. From centre C with radius equal to a, describe an arc of a circle. This must touch or cut AB, or the triangle cannot be constructed.

First, let it touch AB in B, as in (1); then CAB is the triangle required. It is right-angled at B. (Euc. III. 18.)

Secondly, let it cut AB, but let radius a be less than CA or b. It will then cut AB twice—in B and B' suppose, as in (2); and two triangles CAB and CAB' are formed, each of which has three of its parts equal to A, a, b, respectively. Also, writing B' for angle $AB'C$, since $CB = CB'$,

$$\therefore \sin B = \sin CB'B = \sin B'.$$

Hence in this case the geometrical construction, like the trigonometrical solution, gives us two triangles.

Thirdly, let its radius be greater than CA or b. It will then cut AB but once—in B, suppose. The point corresponding to B' will now be in BA produced. Hence here, as in the solution, there is but one triangle CAB obtained.

96. The preceding articles may be applied as in the following examples.

(1.) Prove that $\cot B - \cot A = \dfrac{a^2-b^2}{ab} \operatorname{cosec} C$.

$$\cot B - \cot A = \frac{\sin(A-B)}{\sin A \sin B}$$
$$= \frac{\sin(A+B)\sin(A-B)}{\sin A \sin B \sin C}$$
$$= \frac{\sin^2 A - \sin^2 B}{\sin A \sin B} \operatorname{cosec} C$$
$$= \left(\frac{\sin A}{\sin B} - \frac{\sin B}{\sin A}\right) \operatorname{cosec} C$$
$$= \left(\frac{a}{b} - \frac{b}{a}\right) \operatorname{cosec} C$$
$$= \frac{a^2-b^2}{ab} \operatorname{cosec} C.$$

(2.) Given, in the triangle ABC, $a=\frac{3}{4}$ foot, $b=1$ foot, $c=1\frac{1}{2}$ feet; find the number of degrees in the least angle of the triangle.

Given $\cos 26° 22' = \cdot 8958$.

The least side being a, by Euc. I. 18 the least angle is A. And

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{1+(\frac{3}{2})^2-(\frac{3}{4})^2}{2 \,.\, 1 \,.\, \frac{3}{2}}$$
$$= \frac{1+\frac{9}{4}-\frac{9}{16}}{3} = \frac{43}{48} = \cdot 8958;$$
$$\therefore \cos A = \cos 26° 23';$$
$$\therefore A = 26° 23'.$$

(3.) When $a=4$, $b=5$, $C=125° 6'$; find c.

Given $\cos 54° 54' = \cdot 575$.

$$c^2 = a^2+b^2-2ab\cos C$$
$$= 16+25+2 \,.\, 4 \,.\, 5 \cos(180°-C)$$
$$= 41 + 40 \times \frac{575}{1000} = 64;$$
$$\therefore c = 8.$$

(4.) Given $\sin B = \cdot 25$, $a=5$, $b=2\cdot 5$; find A.

$$\sin A = \frac{a}{b}\sin B$$
$$= \frac{5}{\frac{5}{2}} \times \frac{1}{4} = \frac{1}{2}$$
$$= \sin 30° \text{ or } \sin 150°.$$

This is the ambiguous case, since A is greater than B;

$$\therefore A = 30° \text{ or } 150°.$$

EXAMPLES.

Prove the formulæ in 1—27.

1. $b \sin A - a \sin B = 0.$

2. $b \sin A + c \sin A = a (\sin B + \sin C).$

3. $a (\sin B - \sin C) + b (\sin C - \sin A) + c (\sin A - \sin B) = 0.$

4. $b (\sin A - \sin C) - (a-c) \sin B = 0.$

5. $\dfrac{a}{b} = \sqrt{\left\{\dfrac{(1-\cos A)\ (1+\cos A)}{(1-\cos B)\ (1+\cos B)}\right\}}.$

6. $2a \operatorname{cosec} A = b \operatorname{cosec} B + c \operatorname{cosec} C.$

7. $a \sec A + b \sec B = c \sec A \sec B.$

8. $(a+b) \cos C + (b+c) \cos A + (c+a) \cos B = a+b+c.$

9. $\cot B + \cot A = \dfrac{c \operatorname{cosec} A}{b}.$

10. $a^2+b^2+c^2 = 2ab \cos C + 2ac \cos B + 2bc \cos A.$

11. $\cos (B+C) = \dfrac{a^2-b^2-c^2}{2bc}.$

12. $a (b \cos C - c \cos B) = b^2-c^2.$

13. $a^2 \sin A + ab \sin B + ac \sin C = (a^2+b^2+c^2) \sin A.$

14. $a (\cos B \cos C + \cos A) = b (\cos A \cos C + \cos B).$

15. $b (\cos B + \cos A \cos C) = c (\cos C + \cos A \cos B).$

16. $a^2+2ab \sin (C-30^\circ) = c^2+2bc \sin (A-30^\circ).$

17. $b \cos B + c \cos C = a \cos (B-C).$

18. $a \cos A + b \cos B + c \cos C = 2a \sin B \sin C.$

19. $\cos A + \cos B + \cos C = 1 + \dfrac{2a \sin B \sin C}{a+b+c}.$

20. $\tan \dfrac{A+B}{2} = \cot \dfrac{C}{2}.$

21. $\sin \dfrac{A-B}{2} = \dfrac{a-b}{c} \cos \dfrac{C}{2}.$

22. $\cos \dfrac{A-B}{2} = \dfrac{a+b}{c} \sin \dfrac{C}{2}.$

23. $\dfrac{\sin \left(\dfrac{A}{2} + B\right)}{\sin \dfrac{A}{2}} = \dfrac{b+c}{a}.$

24. If $A : B : C = 2 : 3 : 4$, then

$$\cos\frac{A}{2} = \frac{a+c}{2b}.$$

25. $\sin(A-B) = \dfrac{a^2-b^2}{c^2}\sin C.$

26. The perpendicular from A upon a

$$= \frac{b^2\sin C + c^2\sin B}{b+c}.$$

27. $$\frac{a^2-b^2}{c^2}\sin C + \frac{b^2-c^2}{a^2}\sin A + \frac{c^2-a^2}{b^2}\sin B$$
$$+ 4\sin\frac{A-B}{2}\sin\frac{B-C}{2}\sin\frac{C-A}{2} = 0.$$

28. If a, b, c are $1\frac{1}{2}$ feet, $1\frac{3}{5}$ feet, and 2 feet respectively, how many degrees are there in the angle C?

29. In the triangle ABC, $a = 7$, $b = 2\sqrt{7}$, $c = 3\sqrt{7}$; what is angle A?

30. Given $b = 3$, $c = 5$, $A = 120°$; required a.

31. Two sides of a triangle are $3\sqrt{6}$ yards and $3(\sqrt{3}+1)$ yards respectively, and include an angle of 45°; what are the remaining angles and side?

32. Given $A = 60°$, $B = 45°$, $a = \sqrt{6}$; find b.

33. Given $B = 45°$, $a = 3\sqrt{2}$, $b = 2\sqrt{3}$; required A.

34. Given $A = 30°$, $a = \sqrt{2}$, $b = 2$; find B.

35. Given $A = 45°$, $a = 2$, $b = \sqrt{2}$; find B.

36. Given $A = 30°$, $a = 2$, $b = 5$; solve the triangle.

37. Given $A = 18°$, $a = 2$, $b = 2+\sqrt{20}$; solve the triangle. Then state why the solution is not ambiguous.

38. When a, b, c are 3, 4, 5 feet, respectively; find the greatest and least angles of ABC.

Given $\cos 36° 52' = \cdot 8$.

39. The sides of a triangle measure 5 furlongs, 1 mile, and 1 mile 1 furlong, respectively. Determine the angle subtended by the distance 1 mile.

Given $\cos 62° 11' = \cdot 4666$.

40. In ABC, $a = 1$ mile, $b = 1\frac{1}{2}$ mile, and $c = 2$ miles; what is the number of degrees in the greatest angle?

Given $\cos 49° 33' = \cdot 65$.

41. If $a = 5$, $b = 8$, $C = 97° 53' 50''$; find C.
Given $\cos 82° 6' 10'' = \cdot 1375$.

42. If $A = 131° 35'$, $B = 30°$, $b = 5\frac{1}{2}$ furlongs; find a.
Given $\sin 48° 25' = \cdot 75$.

43. When $a = 15$, $b = 8$, $B = 23° 25'$; find A.
Given $\sin 23° 25' = \cdot 4$.
$\sin 48° 25' = \cdot 75$.

44. The sides of a triangle are a, b, and $\sqrt{a^2 + ab + b^2}$; what is the greatest angle?

45. The sides being given, how in general may it be determined whether or not the triangle contains an obtuse angle?

CHAPTER XII.

CERTAIN FORMULÆ FOR SOLUTION OF TRIANGLES ADAPTED TO LOGARITHMIC COMPUTATION.—AREA OF TRIANGLE.—CIRCUMSCRIBED, INSCRIBED, AND ESCRIBED CIRCLES.

97. "*Adapted to Logarithmic Computation.*" The full force of this phrase is to be seen only after logarithms themselves have been studied; but some explanation will be required for present purposes.

The use of logarithms is of great advantage in calculations which involve arithmetical operations of the following kinds only: multiplication, division, involution, and evolution. *To adapt an expression to logarithmic computation* is so to transform it, that its value may be calculated by the means alone of one or more of these operations.

For instance, we adapt the following

$$1+\tan^2 A,\ \sqrt{b^2-c^2},\ a^2-a^2\cos^2 A,\ \frac{a^2-(b-c)^2}{4bc},$$

by transforming and writing them thus:

$$\sec^2 A,\ \sqrt{(b+c)\ (b-c)},\ a^2\sin^2 A,\ \frac{(a-b+c)(a+b-c)}{4bc}$$

It will be observed that formula (2) of Chapter XI. is adapted to logarithmic computation, while (3) and (4) are not. It is a principal object of this Chapter to replace (3) by other formulæ deduced from it which are so adapted. In a subsequent chapter, (4) will be somewhat similarly treated.

98. Before proceeding, certain abbreviations require careful attention.

Definition.—The *perimeter* of any rectilineal figure is the sum of its sides.

It is usual to write s for *semi-perimeter* of the triangle ABC, so that

$$s = \frac{a+b+c}{2} \text{ and } 2s = a+b+c;$$

also $\quad 2s-2a = a+b+c-2a, \therefore 2(s-a) = b+c-a;$

$\quad\quad 2s-2b = a+b+c-2b, \therefore 2(s-b) = a+c-b;$

and $\quad 2s-2c = a+b+c-2c, \therefore 2(s-c) = a+b-c.$

We obviously deduce that

$$s-a = \frac{b+c-a}{2},\ s-b = \frac{a+c-b}{2},\ s-c = \frac{a+b-c}{2}.$$

99. *To express* $\sin\frac{A}{2}$ *and* $\cos\frac{A}{2}$ *in terms of the sides of the triangle* ABC.

(i.) Since $\cos A = 1-2\sin^2\frac{A}{2}$;

transposing,

$$2\sin^2\frac{A}{2} = 1-\cos A = 1-\frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{2bc-b^2-c^2+a^2}{2bc}$$

$$= \frac{a^2-(b^2-2bc+c^2)}{2bc};$$

$$\therefore \sin^2\frac{A}{2} = \frac{a^2-(b-c)^2}{4bc} = \frac{(a-b+c)(a+b-c)}{4bc}$$

$$= \frac{2(s-b)\times 2(s-c)}{4bc}$$

$$= \frac{(s-b)(s-c)}{bc};$$

$$\therefore \sin\frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}}.$$

(ii.) Since $\cos A = 2\cos^2\frac{A}{2} - 1$;

$$\therefore\ 2\cos^2\frac{A}{2} = 1 + \cos A = 1 + \frac{b^2+c^2-a^2}{2bc};$$

$$\therefore\ \cos^2\frac{A}{2} = \frac{(b+c)^2-a^2}{4bc} = \frac{2s\,.\,2\,(s-a)}{4bc};$$

$$\therefore\ \cos\frac{A}{2} = \sqrt{\left\{\frac{s\,(s-a)}{bc}\right\}}.$$

Cor.—Since $2\sin\frac{A}{2}\cos\frac{A}{2} = 2\sqrt{\left\{\frac{(s-b)(s-c)}{bc}\times\frac{s(s-a)}{bc}\right\}}$;

$$\therefore\ \sin A = \frac{2}{bc}\sqrt{s\,(s-a)\,(s-b)\,(s-c)}.$$

100. *Expressions for other functions of* $\frac{A}{2}$.

These easily follow from those of the last article. For instance,

$$\because\ \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \sqrt{\left\{\frac{s-b)\,(s-c)}{bc}\times\frac{bc}{s\,(s-a)}\right\}};$$

$$\therefore\ \tan\frac{A}{2} = \sqrt{\left\{\frac{(s-b)\,(s-c)}{s\,(s-a)}\right\}}.$$

And in the same way,

$$\cot\frac{A}{2} = \sqrt{\left\{\frac{s\,(s-a)}{(s-b)\,(s-c)}\right\}}.$$

The expressions for $\sec\frac{A}{2}$ and $\operatorname{cosec}\frac{A}{2}$ are of course inversions of those of $\cos\frac{A}{2}$ and $\sin\frac{A}{2}$.

101. In articles 99, 100, A is an angle of a triangle. Therefore always

$$A \text{ is } < 180^\circ, \text{ and } \frac{A}{2} < 90^\circ.$$

It is inferred that the foregoing radicals are not ambiguous in sign, but are always positive. On examination they will also be found adapted to logarithmic computation. They are therefore suited for ordinary solution of triangles, which requires the use of logarithms.

102. *Expressions for area of a triangle.*

(i.) *Given two sides and the included angle.*—Construct as in Art. 88; and write Δ for area of triangle ABC. Then, by Euc. I. 41, Δ is half the area of the rectangle on the same base and of equal altitude;

$$\therefore\ \Delta = \frac{1}{2} AB \,.\, CD = \frac{1}{2} AB \,.\, BC \sin CBD,$$

or

$$\Delta = \frac{1}{2} ac \sin B.$$

(ii.) *Given three sides.*—Applying (i.),

$$\Delta = \frac{ac}{2} \sin B = \frac{ac}{2} \times \frac{2}{ac} \sqrt{s(s-a)(s-b)(s-c)};$$

$$\therefore\ \Delta = \sqrt{s(s-a)(s-b)(s-c)}.$$

(iii.) *Given one side and two adjacent angles.*—Since

$$\frac{c}{a} = \frac{\sin C}{\sin A}, \quad \therefore\ c = a \frac{\sin C}{\sin A}.$$

Substituting for c in (i.),

$$\Delta = \frac{1}{2} a \sin B \times a \frac{\sin C}{\sin A}$$

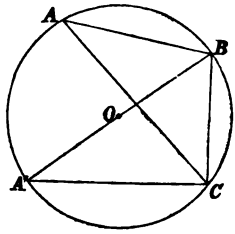
$$= \frac{a^2}{2} \cdot \frac{\sin B \sin C}{\sin A} = \frac{a^2}{2} \cdot \frac{\sin B \sin C}{\sin (B+C)};$$

$$\therefore\ \Delta = \frac{a^2}{2} \sin B \sin C \operatorname{cosec} (B+C).$$

103. *Radius of circumscribing circle.*

(i.) *In terms of one angle and the opposite side.*

Let O be the centre of the circle circumscribing the triangle ABC. (Euc. IV. 5.) Draw a diameter BOA', and join $A'C$.



Then by Euc. III. 21, $\angle A = \angle A'$, because they are in the same segment; also by Euc. III. 31, BCA' in a semicircle is a right angle, and therefore BCA' is a right-angled triangle.

Wherefore, writing R for radius of circumscribing circle,

$$\sin A = \sin A' = \frac{BC}{BA'} = \frac{a}{2R};$$

$$\therefore\ 2R \sin A = a;$$

$$\therefore\ R = \frac{a}{2 \sin A}.$$

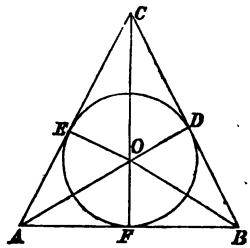
(ii.) *In terms of three sides and area.*

Because $\Delta = \dfrac{bc \sin A}{2}$, or $\sin A = \dfrac{2\Delta}{bc}$;

therefore, substituting for $\sin A$ in formula just found,

$$R = \frac{abc}{4\Delta}, \text{ or } R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

104. *Radius of inscribed circle in terms of sides and area of triangle.*



Inscribe the circle DEF in the triangle ABC touching it in D, E, F, by Euc. IV. 4; and let O be its centre. Write r for radius of it; then

$$r = OD = OE = OF.$$

Also write Δ for area of ABC; then

$$\Delta = \text{area } BOC + \text{area } COA + \text{area } AOB$$

$$= \tfrac{1}{2}ar + \tfrac{1}{2}br + \tfrac{1}{2}cr;$$

for OD, OE, OF are perpendiculars to BC, CA, AB, (Euc. IV. 4);

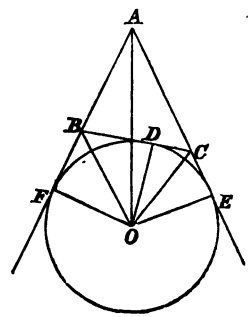
$$\therefore\ \Delta = \frac{a+b+c}{2} r = s \,.\, r;$$

$$\therefore\ r = \frac{\Delta}{s}, \text{ or } r = \sqrt{\left\{\frac{(s-a)\ (s-b)\ (s-c)}{s}\right\}}.$$

105. *Def.*—An *escribed circle* of a triangle touches one of its sides and the others produced.

In Euc. IV. 4. it is shown that the point in which the lines meet which bisect the angles A, B, C is the centre of the inscribed circle. If AB, AC be produced, the same mode of proof will show that the centre of the escribed circle touching a is the intersection of the straight lines AO, BO, CO, which respectively bisect A and the exterior angles adjacent to B and C formed by the produced sides.

106. *Radius of escribed circle in terms of sides and area of triangle.*



Describe the circle DEF touching the side BC of the triangle ABC and AC, AB produced in D, E, F respectively. Write r_a for its radius; then

$$r_a = OD = OE = OF.$$

Also write Δ for area of triangle ABC; then

$$\text{area } OBAC = \text{area } OBC + \Delta,$$

$$\text{and area } OBAC = \text{area } OAC + \text{area } OAB;$$

$$\therefore\ \Delta + \text{area } OBC = \text{area } OAC + \text{area } OAB,$$

or $$\Delta + \tfrac{1}{2} a r_a = \tfrac{1}{2} b r_a + \tfrac{1}{2} c r_a,$$

for OD, OE, OF are perpendiculars to BC, AC produced, and AB produced.

$$\therefore\ \Delta = \frac{b}{2} r_a + \frac{c}{2} r_a - \frac{a}{2} r_a = \frac{b+c-a}{2} r_a = (s-a)\, r_a$$

$$\therefore\ r_a = \frac{\Delta}{s-a}, \text{ or } r_a = \sqrt{\left\{\frac{s\ (s-b)\ (s-c)}{s-a}\right\}}$$

With a similar notation,

$$r_b = \frac{\Delta}{s-b}, \quad r_c = \frac{\Delta}{s-c}.$$

107.—(1.) Prove the formula $c = r\left(\cot\frac{A}{2} + \cot\frac{B}{2}\right)$.

$$r\left(\cot\frac{A}{2} + \cot\frac{B}{2}\right) = r\left(\frac{\sqrt{s(s-a)}}{\sqrt{(s-b)(s-c)}} + \frac{\sqrt{s(s-b)}}{\sqrt{(s-a)(s-c)}}\right)$$

$$= r\,\frac{(s-a)\sqrt{s} + (s-b)\sqrt{s}}{\sqrt{(s-a)(s-b)(s-c)}}$$

$$= r\,\frac{\sqrt{s(s-a+s-b)}}{\sqrt{(s-a)(s-b)(s-c)}}$$

$$= \frac{\sqrt{(s-a)(s-b)(s-c)}}{\sqrt{s}} \times \frac{\sqrt{s}\,.\,c}{\sqrt{(s-a)(s-b)(s-c)}}$$

$$= c.$$

(2.) Find the area of a triangle whose sides are respectively 2, 3, 4 units of length.

Here $$s = \frac{2+3+4}{2} = \frac{9}{2};$$

$$\therefore\ s-a = \frac{9}{2} - 2 = \frac{5}{2}, \quad s-b = \frac{3}{2}, \quad s-c = \frac{1}{2}.$$

$$\therefore\ \text{required area} = \sqrt{\left\{\frac{9}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\right\}}$$

$$= \sqrt{\left\{\frac{9\times 15}{16}\right\}} = \frac{3}{4}\sqrt{15} = 2{\cdot}9 \text{ nearly.}$$

EXAMPLES.

Prove the truth of the statements in 1—25.

1. $\cot\frac{A}{2}\cot\frac{B}{2} = \frac{a+b+c}{a+b-c}$.

2. $c\sin^2\frac{B}{2} + b\sin^2\frac{C}{2} = s-a$.

3. $1-\tan\frac{A}{2}\tan\frac{B}{2} = \frac{2c}{a+b+c}$.

4. Area of triangle $= \sin\frac{A}{2}\sqrt{bc\,.\,s(s-a)}$.

5. $\text{Area} = \dfrac{s \cdot c}{\cot\dfrac{A}{2} + \cot\dfrac{B}{2}}.$

6. $\text{Area} = \dfrac{a^2-b^2}{2} \cdot \dfrac{\sin A \sin B}{\sin(A-B)}.$

7. $r = \dfrac{bc}{2s}\sin A$, and $r_a = \dfrac{ac}{2(s-b)}\sin B.$

8. $\text{Area} = \sqrt{r r_a r_b r_c}.$

9. $4R + s = abc.$

10. $\dfrac{b \text{ vers } A}{a \text{ vers } B} = \dfrac{b+c-a}{a+c-b}.$

11. $a+b+c = 2c\cos\dfrac{A}{2}\cos\dfrac{B}{2}\operatorname{cosec}\dfrac{C}{2}.$

12. $\dfrac{\tan\dfrac{A}{2} + \tan\dfrac{B}{2}}{\tan\dfrac{A}{2} - \tan\dfrac{B}{2}} = \dfrac{c}{a-b}.$

13. $\dfrac{\cot\dfrac{A}{2}}{\cot\dfrac{B}{2} + \cot\dfrac{C}{2}} = \dfrac{s-a}{a}.$

14. $8R = (a+b+c)\sec\dfrac{A}{2}\sec\dfrac{B}{2}\sec\dfrac{C}{2}.$

15. $\text{Area} = s^2\tan\dfrac{A}{2}\tan\dfrac{B}{2}\tan\dfrac{C}{2}.$

16. $\text{Area} = Rr(\sin A + \sin B + \sin C).$

17. $4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2} = r, \quad 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2} = r_a.$

18. $\tan^2\dfrac{A}{2} = \dfrac{r r_a}{r_b r_c}.$

19. $bc\cos^2\dfrac{A}{2} + ac\cos^2\dfrac{B}{2} + ab\cos^2\dfrac{C}{2} = 4(r_a r_b + r_a r_c + r_b r_c).$

20. $\text{Area} = \dfrac{a^2+b^2+c^2}{4(\cot A + \cot B + \cot C)}.$

In 21—25, C is a right angle.

21. $\sec 2A = \dfrac{c^2}{b^2-a^2}$.

22. $\sin^2 \dfrac{A}{2} - \sin^2 \dfrac{B}{2} = \dfrac{a-b}{2c}$.

23. $\tan \dfrac{A}{2} = \dfrac{a}{b+c}$.

24. $R+r = \dfrac{a+b}{2}$.

25. Area $= s(s-c)$.

26. If d be the perpendicular from C upon c, show that $\sin C = \dfrac{cd}{ab}$.

27. Given A, B, C, and R the radius of circumscribing circle; find a, b, c.

28. The sides of a triangle are 7, 8, 9; required the area.

29. The side of an equilateral triangle is 20; apply the formula $\Delta = \frac{1}{2} bc \sin A$ to find its area.

30. Two sides of a triangle are 3 and 12 respectively, and contain an angle of 30°; what will be the hypothenuse of a right-angled isosceles triangle of equal area?

31. The sides of a parallelogram, including an angle of 60°, are 20 and 30; and the sides of another, including an angle of 120°, are 30 and 40. Compare their areas.

32. Show, from the expression for area of a triangle in terms of its sides, that of all triangles which have a given base, and the sum of their sides given, the isosceles triangle has the greatest area.

33. Show that the sum of the areas of the triangles which fulfil the conditions of the problem in the ambiguous case is $b^2 \sin A \cos A$.

34. Prove that the distance of the centre of the inscribed circle from A is equal to $\dfrac{2bc}{a+b+c} \cos \dfrac{A}{2}$.

35. If $C = \dfrac{\pi}{2}$ in the triangle ABC, then

$$\tan^{-1} \frac{a}{b+c} + \tan^{-1} \frac{b}{a+c} = \frac{\pi}{4}.$$

36. In a plane triangle the lengths of two sides are 169 and 126 yards respectively, and the included angle $= 2\sin^{-1}\frac{33}{65}$; find the third side and the area.

37. If, in the ambiguous case, c, c' be the two values of the third side, and the given angle be 45°, show that the angle between the two positions of the side opposite is $\cos^{-1}\frac{2cc'}{c^2+c'^2}$.

38. Show that

$$a\cos A + b\cos B + c\cos C = 4R\sin A\sin B\sin C.$$

39. Prove (i.) $-\frac{1}{r}+\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=0.$

(ii.) $-r+r_a+r_b+r_c=4R.$

40. If α, β, γ be the angles which the sides subtend at the centre of the inscribed circle,

$$4\sin\alpha\sin\beta\sin\gamma = \sin A + \sin B + \sin C.$$

CHAPTER XIII.

LOGARITHMS.

108. Of Logarithms it is possible to construct any number of different *systems*. Each of these would have what is called a *base* peculiar to itself; and would provide a logarithm for every number. The relation in which base, logarithm, and number stand to one another in any system may be thus shown:

If a, x, N, be numbers such that

$$a^x = N;$$

then in that system of logarithms whose base is a, the index x is the logarithm of the number N; or, as it is usual to express it, x is the logarithm of N to the base a. Hence the

Definition.—The logarithm of a number to a given base is the index of the power to which the base must be raised to be equal to the given number.

For brevity's sake we write $\log_a N$ for the words *the logarithm of N to the base a.*

So that if $\quad a^x = N, \quad$ then $\quad x = \log_a N;$

or if $\quad x = \log_a N, \quad$ then $a^x = N.$

109. *The logarithm of the product of any number of factors is equal to the sum of the logarithms of the factors.*

Let there be n factors of a number, namely, N_1, N_2, N_3,.....N_n. Then it is required to prove that

$$\log N_1 . N_2 . N_3 \ldots . N_n$$
$$= \log N_1 + \log N_2 + \log N_3 + \ldots + \log N_n.$$

Taking a as base, let

$$x = \log N_1, \text{ and } y = \log N_2,$$

so that $$N_1 = a^x, \text{ and } N_2 = a^y.$$

Then $$N_1 N_2 = a^x \,.\, a^y = a^{x+y};$$

therefore $$\log N_1 N_2 = x+y = \log N_1 + \log N_2.$$

Applying this result, it follows that

$$\begin{aligned} \log N_1 N_2 N_3 &= \log N_1 N_2 + \log N_3 \\ &= \log N_1 + \log N_2 + \log N_3, \\ \log N_1 N_2 N_3 N_4 &= \log N_1 N_2 + \log N_3 N_4 \\ &= \log N_1 + \log N_2 + \log N_3 + \log N_4, \end{aligned}$$

and so on; which makes it clear that

$$\begin{aligned} &\log N_1 N_2 N_3 \ldots\ldots N_n \\ &\quad = \log N_1 + \log N_2 + \log N_3 + \ldots\ldots + \log N_n. \end{aligned}$$

110. *The logarithm of the quotient of two numbers is the difference of the logarithms of the numbers.*

Let $\frac{N}{N'}$ be the quotient. It is to be proved that

$$\log \frac{N}{N'} = \log N - \log N'.$$

Taking a as base, let

$$x = \log N, \text{ and } y = \log N',$$

so that $$N = a^x, \text{ and } N' = a^y.$$

Then $$\frac{N}{N'} = \frac{a^x}{a^y} = a^{x-y};$$

therefore $$\log \frac{N}{N'} = x - y = \log N - \log N'.$$

111. *The logarithm of a power of any number is the product of the logarithm of the number and the index of the power.*

Let N^p be any power of any number; required to prove that $$\log N^p = p \log N.$$

Taking a for base, let

$$x = \log N, \text{ then } a^x = N;$$

therefore $$N^p = (a^x)^p = a^{px};$$

therefore $$\log N^p = px = p \log N.$$

COR.—This result is true for all values of p, so that

$$\log \sqrt[r]{N} = \log N^{\frac{1}{r}} = \frac{1}{r} \log N.$$

112. It will hereafter be made clear how by means of logarithms we may substitute

by Art. 109, addition for multiplication,
by Art. 110, subtraction for division,
by Art. 111, multiplication for involution,
and by Art. 111 *Cor.*, division for evolution.

Some conception may hence be derived of the very great utility of logarithms, when it is considered that, in practical applications of Trigonometry and the kindred Mathematics, the numbers or fractions involved are very commonly large. The three formulæ already obtained are the leading formulæ of Logarithms, and merit close attention.

113. Although, as before stated, it is possible to construct any number of systems of logarithms; yet two only are ever put to practical use, and the first of these will merely require mention at present. They are

(i.) *Napierian Logarithms*, so called because employed by Lord Napier, the inventor of logarithms. Their base is 2·7182818..., which is written e or ϵ.

(2.) *Briggs' or the Common Logarithms*, of which the base is 10. This base was first proposed by Professor Briggs of Oxford, and for numerical calculation will be shown to possess important advantages over any other base. The base 10 is always understood in numerical logarithms, when no other is stated.

114. Notice that, a being any base,

$$a^0 = 1;$$

hence log 1 to any base $= 0$.

Also $$a^1 = a,$$
therefore log a to base $a = 1$; so that in common logarithms $\log 10 = 1$.

EXERCISE VII.—(1.) Resolve log 75 into elementary logarithms.

$$\begin{aligned}\log 75 &= \log 25 \times 3 = \log 25 + \log 3\\ &= \log \frac{10^2}{2^2} + \log 3\\ &= \log 10^2 - \log 2^2 + \log 3\\ &= 2 - 2\log 2 + \log 3.\end{aligned}$$

(2.) Simplify $\log \dfrac{a\sqrt{b}}{\sqrt[3]{abc}}$.

$$\begin{aligned}\log \frac{a\sqrt{b}}{\sqrt[3]{abc}} &= \log \frac{ab^{\frac{1}{2}}}{a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}}\\ &= \log ab^{\frac{1}{2}} - \log a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}\\ &= \log a + \log b^{\frac{1}{2}} - \log a^{\frac{1}{3}} - \log b^{\frac{1}{3}} - \log c^{\frac{1}{3}}\\ &= \log a + \tfrac{1}{2}\log b - \tfrac{1}{3}\log a - \tfrac{1}{3}\log b - \tfrac{1}{3}\log c.\\ &= \tfrac{2}{3}\log a + \tfrac{1}{6}\log b - \tfrac{1}{3}\log c.\end{aligned}$$

Simplify in a similar manner the expressions in 3—16.

3. $\log xyz$. 4. $\log 20$. 5. $\log \dfrac{ab^2c^3}{4x^4y^5}$.

6. $\log \dfrac{1}{4}$. 7. $\log 150$. 8. $\log 6ab\sqrt{bcd}$.

9. $\log \dfrac{\sqrt[4]{48}}{81}$. 10. $\log \dfrac{\sqrt[3]{9\sqrt{8}}}{\sqrt[4]{5\sqrt[3]{4}}}$. 11. $\log c\dfrac{\sin B}{\sin C}$.

12. $\log \frac{1}{2}ac \sin B$. 13. $\log \sqrt{(b+c)(b-c)}$.

14. $\log \sqrt{\left\{\dfrac{(s-b)(s-c)}{bc}\right\}}$. 15. $\log \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$.

16. $\log (a+b)\sin\dfrac{C}{2}\sec\phi$.

115. Common logarithms alone will be treated of in the remainder of this chapter.

Definitions.—(1.) The *characteristic* of the logarithm of any number is the integral portion of it.

(2.) The *mantissa* of the logarithm of any number is the decimal portion of it.

For instance, it is known that
$$\log 3145 = 3{\cdot}4976206;$$

3 is the characteristic therefore, and ·4976206 the mantissa of log 3145.

116. The following remarks may render more evident the reasoning of the important proposition which comes after them.

The least integer consisting of n digits is 10^{n-1}; for instance, the least integer of four digits is 1000, which is 10^3. Any other integer consisting of n digits, with or without a decimal fraction attached, is between 10^{n-1} and 10^n; thus 67642, which contains five digits, is between 10,000 and 100,000, that is, between 10^4 and 10^5; and 123456·421 is between 100,000 and 1,000,000, or between 10^5 and 10^6.

Again, the least decimal fraction which has n ciphers *immediately* after the decimal point is $10^{-(n+1)}$; for instance, the least decimal with three ciphers thus placed is $\cdot 0001 = \frac{1}{10{,}000} = \frac{1}{10^4} = 10^{-4}$. Any other decimal fraction, with n ciphers immediately following the point, is between $10^{-(n+1)}$ and 10^{-n}; thus ·0005 is between ·0001 and ·001, that is, between 10^{-4} and 10^{-3}; and so in all similar cases.

The student, if acquainted with the theory of Notation in Algebra, may prove generally the above statements; if not, he may without difficulty satisfy himself of their truth by simple reflection.

117. *To determine the characteristic of the logarithm of any number.*

This can always be done by *inspection* of the number itself, as will be shown.

(i.) *Let the number be greater than unity*, and consist of an integer of n digits with or without a decimal fraction. It is then either 10^{n-1}, or between 10^{n-1} and 10^n. Its logarithm is therefore either $n-1$ or between $n-1$ and n. Therefore the characteristic of its logarithm is $n-1$. Hence the rule,—

The characteristic of the logarithm of a number greater

than unity is less by one than the number of digits in the integral part of the number.

(ii.) *Let the number be a decimal fraction.* Since $\log 1 = 0$, the logarithms of all decimal fractions will be negative. It is found convenient in practice to have the characteristic only of these logarithms negative, and the mantissa positive. Remembering this, we shape our reasoning as follows:—

Let there be n ciphers immediately after the point. It is then either $10^{-(n+1)}$, or between $10^{-(n+1)}$ and 10^{-n}. Its logarithm is therefore either $-(n+1)$, or, preserving the mantissa positive, $-(n+1)+$a fraction. Therefore the characteristic of its logarithm is $-(n+1)$. Hence the rule,—

The characteristic of the logarithm of a decimal fraction is greater by one than the number of ciphers which immediately *follow the point, and is negative.*

118. It would perhaps assist the memory to incorporate the two rules given above into one, thus:—

If the number be greater than unity, take the number of its integral digits positively; *if it be a decimal fraction, take the number of ciphers which immediately follow the point* negatively; *subtract one in each case for the characteristic of its logarithm.*

119. In the logarithms of decimal fractions the minus sign is placed *above* the characteristic to denote that it applies only to the characteristic, and not to the mantissa, which, as before said, is always preserved positive.

120. *Significant and non-significant digits.*

Take the number

56·0802,

and either multiply or divide it by any power of 10:—for instance, multiply it by 10^3, 10^4, and by 10^6; and divide it by 10^2 and 10^5; and we obtain

(1) 56080·2, (2) 560802·, (3) 56080200·, (4) ·560802, (5) ·000560802;

placing the decimal point in (2) and (3), where ordinarily it would be merely understood.

In all the above numbers, the digits 5, 6, 0, 8, 0, 2 appear in order, while in two of them, (3) and (5), some additional ciphers occur. These latter are merely used to fix the position of the decimal point, when by the multiplication or division it is moved away from the former digits to the right or left.

The digits which *always* occur in such sets of numbers as the above are called *significant* digits; whilst the additional ciphers which are used occasionally, and merely to find the position of the decimal point, are called *non-significant* digits.

It will appear from instances such as the above that, N being any number,

$$N, \quad N \times 10^n, \quad \text{and} \quad N \div 10^n,$$

where n is a positive integer, all contain the same significant digits in the same order.

121. *All numbers which have the same significant figures in the same order, have also the same mantissa in their logarithms; that is,*

$$\log N, \quad \log N \,.\, 10^n, \quad \text{and} \quad \log \frac{N}{10^n}$$

all have the same mantissa when n is a positive integer.

Since
$$\begin{aligned} \log N \,.\, 10^n &= \log N + \log 10^n \\ &= \log N + n \log 10 \\ &= \log N + n \quad \ldots\ldots\ldots\ldots\ldots\ldots (1), \end{aligned}$$

and
$$\begin{aligned} \log \frac{N}{10^n} &= \log N - \log 10^n \\ &= \log N - n \log 10 \\ &= \log N - n \quad \ldots\ldots\ldots\ldots\ldots\ldots (2); \end{aligned}$$

and the decimal part of $\log N$ cannot be altered either by adding the whole number n as in (1), or subtracting it as in (2). Therefore

$$\log N, \quad \log N \,.\, 10^n, \quad \text{and} \quad \log \frac{N}{10^n}$$

have the same mantissa.

122. Examples will now follow on the foregoing.

(1.) Find the logarithm of $16\sqrt[5]{16}$ to the base $\sqrt[3]{2}$.

Let x = required logarithm. Then, by definition of a logarithm,

$$(\sqrt[3]{2})^x = 16\sqrt[5]{16} = 16\sqrt[5]{2^4} = 2^4 \times 2^{\frac{4}{5}};$$

$$\therefore \quad 2^{\frac{x}{3}} = 2^{4+\frac{4}{5}} = 2^{\frac{24}{5}};$$

$$\therefore \quad \frac{x}{3} = \frac{24}{5};$$

$$\therefore \quad x = \frac{72}{5} = 14{\cdot}4 = \text{required logarithm.}$$

(2.) Given $\log 2 = {\cdot}30103$ and $\log 3 = {\cdot}4771213$, find to three places of decimals the value of x in the equation

$$3^{2x} = 45 \,.\, 2^{x-3}.$$

Since $\quad \log 3^{2x} = \log (45 \,.\, 2^{x-3})$;

$$\therefore \quad 2x \log 3 = \log 3^2 \times 5 + \log 2^{x-3}$$

$$= \log 3^2 + \log \frac{10}{2} + (x-3) \log 2$$

$$= 2 \log 3 + 1 - \log 2 + (x-3) \log 2;$$

transposing, $x(2 \log 3 - \log 2) = 1 + 2 \log 3 - 4 \log 2$;

$$\therefore \quad x = \frac{1 + {\cdot}9542426 - 1{\cdot}20412}{{\cdot}9542426 - {\cdot}30103}$$

$$= \frac{{\cdot}7501226}{{\cdot}6532126},$$

or $\quad x = 1{\cdot}148 \ldots.$

(3.) Find $\log 18$ and $\log {\cdot}045$, given that

$$\log 3 = {\cdot}477121, \quad \log 2{\cdot}5 = {\cdot}397940.$$

In order to find $\log 18$, $\log 2$ will be required; and $\log 2$ may be obtained from $\log 25$, which is known from $\log 2{\cdot}5$.

$$\log 25 = \log \frac{100}{4} = \log 10^2 - \log 2^2 = 2 - 2 \log 2;$$

transposing, $\quad 2 \log 2 = 2 - \log 25$

$$= 2 - 1{\cdot}397940$$

$$= {\cdot}60206;$$

$$\therefore \quad \log 2 = {\cdot}30103.$$

Then $\quad \log 18 = \log 3^2 + \log 2$

$$= 2 \log 3 + \log 2$$

$$= {\cdot}954242 + {\cdot}30103,$$

or $\quad \log 18 = 1{\cdot}255272.$

Also $$\log 45 = \log 3^2 + \log 5$$
$$= 2\log 3 + \log \frac{10}{2}$$
$$= 2\log 3 + 1 - \log 2;$$
or $$\log 45 = 1{\cdot}653212;$$
$$\therefore \quad \log {\cdot}045 = \bar{2}{\cdot}653212.$$

Examples.

In these examples the two following logarithms are frequently "given." To save repetition, their values are placed here:

$$\log 2 = {\cdot}30103, \quad \log 3 = {\cdot}4771213.$$

1. Find, by inspection of the following numbers, the characteristics of their logarithms:—3146, ·3146, 200, ·002, 3·05, ·0000101, 101·01, 1876·4, 1·02.

2. Which are the significant digits in 3060, ·05642, 841, ·0001, 11000, and 11001.

3. Prove $\log (3^5 \times 4^3) = 5\log 3 + 6\log 2$.

4. Why have $\log_{10} 3{\cdot}56$ and $\log_{10} {\cdot}0356$ the same mantissa, and $\log_{10} {\cdot}0356$ and $\log_{10} {\cdot}04$ the same characteristic?

5. Write out the logarithms of ·117, 117, and ·000117.
Given $\log 1{\cdot}17 = {\cdot}06818$.

6. Find $\log_2 256$ and $\log 256$ to base $2\sqrt{2}$.

7. What are the logarithms of 100 and ·001 to base 10, and of 1000 to base ·01?

8. What is the logarithm of $81\sqrt[2]{3}$ to the base $3\sqrt{3}$?

9. Find the logarithm of $a^{\frac{p}{q}}$ to the base $a^{\frac{m}{n}}$.

10. Show that $\log_2 N = 3\log_8 N$.

11. Of what number is -5 the logarithm to base 10?

12. Find, without tables and to three places of decimals, the numbers of which 1·5, ·3, and 1·3 are the logarithms in common logarithms.

13. What is the base of the system in which $\log 10 = 2$?

14. Prove that $(\log z)^{\log x} = 1$, when $x = 1$.

15. If the logarithms of numbers in the ordinary tables were all doubled, what would be the base to which they would then be the logarithms of the same numbers as before?

16. How would you transform a system of logarithms from base 8 to base 4?

17. Find $\log_{25} 8$, having given $\log_{10} 2$.

18. Find x from the equation $8^x = 100$, given log 2.

19. Solve $5^{x-1} = 4$, given log 2.

20. Solve $20^{x-2} = \frac{1}{4}$, given log 2.

21. Solve $25^{3-2x} = 2^{x+3}$, given log 2.

22. Solve $\left(\frac{7}{3}\right)^x = 100$, given log 3, and log 7 = ·845098.

23. Solve $2^{3x} \times 6^{3x-5} = 3^{2(x-1)} \times 2^{5x}$, given log 2 and log 3.

24. Given log 2, find log 128, log 125, and log 2500.

25. Find log ·0005, log 196, and $\log \left(\frac{28}{5}\right)^{\frac{3}{4}}$.

Given log 2, and log 7 = ·845098.

26. Find log sin 45°, log tan 60°, log sec 30°; given log 2 and log 3.

27. Find log 3, log ·16 and log 450.

Given log 18 = 1·255272, log 25 = 1·397940.

28. Given log 2 and log 3; find the number of integral digits in

$$\frac{3^{20} \times 5^{15}}{2^{11}}.$$

29. Given log 2 and log 3; find the value of

$$\log \frac{\sqrt[3]{2}\,\sqrt[4]{3}\,\sqrt[5]{4}}{\sqrt[6]{5}\,\sqrt[7]{6}}.$$

30. Given log 2 and log 3; find the logarithms of the mean proportional between

$$\frac{(1{\cdot}25)^{\frac{1}{2}}}{({\cdot}00016)^6} \text{ and } \frac{(1{\cdot}25)^{\frac{3}{2}}}{1000}$$

CHAPTER XIV.

LOGARITHMIC TABLES.

123. *Tables of Logarithms.* From the properties of logarithms treated of in the last chapter will be seen the advantage of having a compilation of the logarithms of the numbers 1, 2, 3, 4, &c. to as high a number as possible. Such a compilation is what is called a "table" of logarithms of the "natural" numbers. We have also other equally useful tables of logarithms, as will appear. Good tables are easily and cheaply procurable, and moreover usually contain a description of the manner in which they are to be used. Hence it will be needless to explain in this treatise the mere manipulation of a copy of them, if the student furnish himself with one; and more needless if he do not. We shall confine ourselves therefore to explaining the simpler and more useful principles of the subject.

124. *Advantage of Briggs', or the Common Logarithms.* A table of common logarithms has the great advantage of being *very much less bulky* than a table computed to any other base than 10; and on two accounts:

(1.) The characteristics of the common logarithms of numbers may all be determined by inspection of the numbers; and the registering of them is therefore dispensed with.

(2.) Having registered the mantissæ of certain numbers, those of all others having the same significant figures are the same, and therefore do not require a separate statement.

125. *Principle of Proportional Parts.* The common tables give the logarithms of whole numbers from 1 to

100,000; which may be made available for all numbers or decimal fractions whatever with not more than five significant figures. By the "principle of proportional parts," they may also be made available for numbers with more than five significant figures, not with perfect accuracy, but with sufficient accuracy for ordinary practical purposes. As applying to numbers, the principle may be stated as follows: *The change in the logarithm of a number is approximately proportional to the change in the number itself, if it be small in comparison with the magnitude of the number.* As applying to angles and their functions it will be hereafter enunciated. It is of course capable of proof; which, however, is beyond the scope of this treatise.

126. *To find the logarithm of a number of more than five significant digits.*

The characteristic is seen on inspection; and we need therefore only concern ourselves with finding the mantissa.

Place the decimal point of the number after the fifth significant digit of it. The mantissa will not be affected by this proceeding.

Let $N+n$ represent the number as thus altered, N being its integral, and n its decimal part.

Also let $d = \log(N+1) - \log N$,

and $\delta = \log(N+n) - \log N$.

Now d is known from the tables; for $\log N$ and $\log(N+1)$ are consecutive logarithms therein; and δ is required.

By the principle of proportional parts,

$$d : \delta :: 1 : n;$$

therefore $$\delta = nd,$$

and is known. And

since $$\log(N+n) - \log N = \delta,$$

therefore $$\log(N+n) = \log N + nd,$$

and is also known; and its mantissa is that which is required.

127. *Given a logarithm which is not in the tables, to find the number whose logarithm it is.*

With the notation of the previous article, evidently $N+n$ is first required, while N, d, δ are known from the tables and the given logarithm. Now, as before,

$$d : \delta :: 1 : n;$$

therefore
$$n = \frac{\delta}{d},$$

and
$$N+n = N + \frac{\delta}{d},$$

which gives a number with the same significant digits as the one required. The decimal point may then be placed in accordance with the value of the given characteristic.

128. In that part of the tables in which the numbers N and $N+1$ occur, the value of d would be called the difference for 1; δ therefore is the difference for n; but is usually called the "proportional part" for n.

129. It will be best to exemplify practically the preceding propositions at once.

(1.) Find the logarithm of the number 349·54572.

By the tables, $\log 34954 = 4{\cdot}5045252$;

and $\log 34955 = 4{\cdot}5045388$;

$\therefore$ diff. for 1 = ·0000136.

Wherefore, to find the proportional part for ·572, we have, writing δ for it, by the principle of proportional parts,

$$\cdot 0000136 : \delta :: 1 : \cdot 572;$$

$$\therefore \quad \delta = \cdot 00000136 \times \cdot 572 = \cdot 000077792;$$

or retaining seven places only of the decimal,

$$\delta = \cdot 0000078;$$

$$\begin{aligned} \therefore \quad \log 34954{\cdot}572 &= 4{\cdot}5045252 \\ &\quad + \cdot 0000078 \\ &= 4{\cdot}5045330, \end{aligned}$$

and $\log 349{\cdot}54572 = 2{\cdot}5045330$.

Obs.—When cutting off some of the digits in a decimal fraction, add unity to the last digit retained, if the first of those cut off be 5 or more than 5.

(2.) Find the number whose logarithm is $\bar{4}\cdot8026755$.

Finding in the tables the two mantissæ between which the given mantissa lies, we have

$$\log 63485 = 4\cdot8026711,$$

and

$$\log 63486 = 4\cdot8026780;$$

$$\therefore \text{ diff. for } 1 = \cdot0000069.$$

Again, given logarithm $= 4\cdot8026755$,

$\therefore$ proportional part $= \cdot0000044$.

Wherefore, if n be the decimal fraction corresponding to this proportional part, we have

$$\cdot0000069 : \cdot0000044 :: 1 : n;$$

$$\therefore n = \frac{44}{69} = \cdot643 \ldots.$$

$$\therefore 4\cdot8026755 = \log 63485\cdot643;$$

$\therefore$ required number $= \cdot00063485643$.

130. *Tables of Proportional Parts.*—The multiplication and division of the preceding examples being somewhat tedious, the best tables of logarithms replace them by means of what are called "tables of proportional parts" for each "difference." These tables are easily constructed, and for the differences we have employed would stand thus:—

Diff.	Pr. Part.	
	136	
	1	14
	2	27
	3	41
	4	54
136	5	68
	6	82
	7	95
	8	109
	9	122

Diff.	Pr. Part.	
	69	
	1	7
	2	14
	3	21
	4	28
69	5	35
	6	41
	7	48
	8	55
	9	62

in which the parts 14, 27, &c. are ·1 of 136, ·2 of 136, &c.; and 7, 14, &c. are ·1 of 69, ·2 of 69, &c. The differences and proportional parts as given above are apparently integers; but they are really decimal fractions of seven places, the points and ciphers being omitted for convenience.

As this part of the subject is rather intricate, it is not further entered on here.

131. Using the tables of proportional parts, the working of the foregoing examples would be as follows :—

(1.) To find log 319·54572.

$$\begin{aligned} \log 319{\cdot}54 &= 2{\cdot}5045252 \\ \text{pro. part for } 5 &= \phantom{2{\cdot}504}68 \\ \text{,, ,, } 7 &= \phantom{2{\cdot}5045}9{,}5 \\ \text{,, ,, } 2 &= \phantom{2{\cdot}50453}|27 \\ \therefore \log 319{\cdot}54572 &= 2{\cdot}5045330. \end{aligned}$$

(2.) To find the number whose logarithm is $\bar{4}{\cdot}8026755$.

$$\begin{aligned} \bar{4}{\cdot}8026755 &= \text{given logarithm} \\ \bar{4}{\cdot}8026711 &= \log {\cdot}00063485 \\ \phantom{\bar{4}{\cdot}802671}\underline{44} & \\ \phantom{\bar{4}{\cdot}802671}\underline{41} &= \text{pr. part for } 6 \\ \phantom{\bar{4}{\cdot}80267}\underline{30} & \\ \phantom{\bar{4}{\cdot}80267}\underline{28} &= \text{pr. part for } 4 \\ \phantom{\bar{4}{\cdot}8026}\underline{20} & \\ \phantom{\bar{4}{\cdot}8026}\underline{21} &= \text{pr. part for } 3. \\ \therefore \text{required number} &= {\cdot}00063485643. \end{aligned}$$

132. *Tables of Natural Sines, Cosines, &c.* In some of the previous chapters, numerical expressions have been obtained for the trigonometrical ratios of the angles 15°, 18°, 30°, &c. These expressions with but little difficulty may be turned into decimal fractions, which, however, very seldom terminate. For instance,

$$\sin 18^\circ = \frac{\sqrt{5}-1}{4} = \frac{2{\cdot}236068\ldots-1}{4} = \frac{1{\cdot}236068\ldots}{4}$$
$$= {\cdot}309017\ldots$$
$$\cos 30^\circ = \frac{\sqrt{3}}{2} = \frac{1{\cdot}73205\ldots}{2} = {\cdot}866025\ldots$$
$$\tan 75^\circ = 2+\sqrt{3} = 2+1{\cdot}73205\ldots = 3{\cdot}73205\ldots$$

Decimal expressions may be obtained not only for the angles referred to, but for any angles; and are called the *natural* functions of the angles. Tables of them are compiled for all angles from 0 to 90°, at

intervals of 1′, or sometimes of 20″ or 10″; and are called tables of natural sines, cosines, &c.

In order to use these tables for angles not registered in them, the principle of proportional parts is applied, and may be stated as follows in this case: *The change in any function of an angle is approximately proportional to the change of the angle, if it be small in comparison with the magnitude of the angle.*

It will be remembered, that up to 90°, as the angle increases in magnitude, the sine, tangent, and secant also increase; but the cosine, cotangent, and cosecant decrease. The application of this fact will be seen in the following examples, and must be carefully attended to.

133. (1.) Find the value of sin 36° 4′ 13″; given from the tables, sin 36° 4′ = ·5887262, sin 36° 5′ = ·5889613.

Here difference for 1′ or 60″ = ·0002351,

$$\therefore \text{ proportional part for } 13'' = \frac{13}{60} \text{ of } \cdot 0002351$$

$$= \cdot 0000509;$$

$$\text{and } \sin 36^\circ\, 4' = \cdot 5887262;$$

$$\therefore \sin 36^\circ\, 4'\, 13'' = \cdot 5887771.$$

Obs. 1.—The differences between each pair of consecutive functions are usually given in tables, omitting their decimal points and the ciphers immediately following the points.

Obs. 2.—If in the above example we had been seeking *cos* 36° 4′ 13″, the proportional part for 13″ would have been *subtracted from*, not added to, cos 36° 4′.

(2.) Find the angle whose natural cosine is ·3784241.

Given cos 67° 45′ = ·3786486, diff. for 1′ = 2692.

Now ·3784241 = given cosine,

and ·3786486 = cos 67° 45′,

∴ 2245 = pro. part;

and proportional part of angle corresponding to it is

$$\frac{2245}{2692} \text{ of } 60'' = 50'' \text{ nearly.}$$

Hence required angle = 67° 45′ 50″.

134. *Tables of Logarithmic Sines, Cosines, &c.*

Tables of logarithmic functions simply comprise the

logarithms of all the numbers which form the tables of natural functions; and the method of using them will be no difficulty after what has been said of the other tables. They have one peculiarity, however, which will require to be noticed.

The sines and cosines of all angles less than 90°, the tangents of angles less than 45°, and the cotangents of angles greater than 45°, are all less than unity; and their logarithms are therefore all negative. In order to render the tables of logarithmic functions all positive, which it is found convenient to do, every log function is increased by 10 before being registered. Each log function thus increased is called a "tabular logarithm," which designation is generally represented by the letter L. Thus

$$L \sin A = \log \sin A + 10,$$
$$L \cos A = \log \cos A + 10,$$
$$L \tan A = \log \tan A + 10, \text{ \&c.}$$

This difference between the real log function and the tabular log function must be carefully remembered in applying the tables; as well as the fact, that as the angle increases, the log sine, log tangent, and log secant increase, but the other log functions decrease.

135. *Complementary Logarithms.*—All the mantissæ registered in a table of logarithms are positive; so that before consulting one to find a number corresponding to a given negative logarithm, the logarithm must be transformed so as to have its mantissa positive, and its characteristic alone negative. How this may be done is easily seen from the following reasoning:—

Let $-(c+m)$ be a negative logarithm, of which c is characteristic and m is mantissa. Then

$$-(c+m) = -c-m = -c-1+1-m,$$

or $$-(c+m) = -(c+1)+(1-m);$$

which gives an equivalent for the given negative logarithms having $1-m$ as new positive mantissa and $-(c+1)$ as new characteristic. The following rule is hence evident: *To transform a negative logarithm so that*

its mantissa may be positive, subtract the mantissa from 1 *for a new positive mantissa, and add* 1 *numerically to the characteristic.*

The operation just described may be made useful in other cases of much more frequent occurrence than that mentioned above—in what manner, will be shown after the following explanations.

When a logarithm preceded by the sign — has been transformed so that its mantissa is positive, it is then called a *complementary logarithm;* and is written briefly "colog." These complementary logarithms are always of service in finding the value of an expression consisting of *more than two* logarithms, of which one or more are negative, and not otherwise; and the utility of them at all times depends upon the readiness with which a mantissa can be subtracted from 1. When required, this should be done *at sight* of the mantissa. It is most readily done by going from left-hand to right, and subtracting each digit from 9 except the last, which must be subtracted from 10.

There are various minor points with which the student may most easily become familiar by studying the following examples. He may perhaps require to be reminded that the sign above a characteristic, when it occurs, as much affects the characteristic as if it stood before it, but has no influence whatever over the mantissa. Thus, for instance,

$$\bar{1}\cdot6824468 = -1+\cdot6824468,$$

while

$$-\bar{1}\cdot3421682 = -(-1)+\cdot3421682$$
$$= +1-\cdot3421682.$$

136. In the first of the Examples which follow, a negative logarithm is transformed, so that it may be looked for in the tables.

1. Find $\log \frac{1}{64}$; given $\log 2 = \cdot3010300$.

$$\text{Log}\,\frac{1}{64} = \log 1 - \log 2^6 = 0 - 6\log 2$$
$$= -1\cdot8061800$$
$$= \bar{2}\cdot1938200.$$

Obs. 1.—This last mantissa is best obtained by subtracting successively 8, 0, 6, 1 from 9, and then 8, the last digit except ciphers, from 10.

Obs. 2.—The chief use of complementary logarithms is to change two or more operations of addition and subtraction into one of addition alone. The following example will illustrate :—

(2.) Find the value of $\log \frac{6}{77 \times 945}$.

Given $\log 2 = \cdot 3010300$, $\log 3 = \cdot 4772213$,

$\log 77 = 1\cdot 8864907$, $\log 945 = 2\cdot 9754318$.

$$\begin{aligned}\log \frac{2 \times 3}{77 \times 945} &= \log 2 + \log 3 - \log 77 - \log 945.\\ &= \log 2 + \log 3 + \text{colog } 77 + \text{colog } 945\\ &= \cdot 3010300\\ &+ \cdot 4771213\\ &+ \bar{2}\cdot 8864907\\ &+ \bar{3}\cdot 9754318\\ &= \bar{3}\cdot 6400738.\end{aligned}$$

137. *Relation between logarithms of the same numbers to different bases.*

Let x, y be the logarithms of the same number N referred to a, b respectively as bases; so that a^x and b^y are each equal to N. Then

$$a^x = b^y, \text{ and } a^{\frac{x}{y}} = b\,;$$

therefore $$\frac{x}{y} = \log_a b\,;$$

therefore $$y = \frac{1}{\log_a b}\, x,$$

or $$\log_b N = \frac{1}{\log_a b} \log_a N.$$

Hence the logarithm of a number in one system may be changed to the logarithm of the same number in another system by multiplying it by the reciprocal of the logarithm in the first system of the base of the second.

138. *Def.—The modulus of common logarithms* is the constant multiplier by means of which Napierian logarithms are converted into common logarithms.

Its value is evident from the result of the last article; by which, ϵ being the base of Napier's logarithms,

$$\log_{10} N = \frac{1}{\log_\epsilon 10} \log_\epsilon N.$$

Now $$\log_\epsilon 10 = 2{\cdot}30258\ldots\ldots;$$

therefore $$\frac{1}{\log_\epsilon 10} = \frac{1}{2{\cdot}30258\ldots} = {\cdot}43429\ldots\ldots;$$

which is therefore the value of the modulus.

139. The first of the following examples exhibits the method of dividing by any number a logarithm which has a negative characteristic. The negative characteristic is increased, if necessary, until it becomes a multiple of the divisor; which negative increase is then compensated by a positive one to the mantissa.

(1.) Find $\log ({\cdot}0002)^{\frac{1}{3}}$; given $\log 2 = {\cdot}30103$.

$$\log ({\cdot}0002)^{\frac{1}{3}} = \tfrac{1}{3} \log {\cdot}0002 = \frac{\bar{4}{\cdot}30103}{3}$$

$$= \frac{\bar{6} + 2{\cdot}30103}{3} = \bar{2}{\cdot}76701.$$

(2.) Find the value of the expression

$$\frac{(8100)^{\frac{1}{3}} \times ({\cdot}4)^3}{360 \times ({\cdot}016)^{\frac{1}{5}}};$$

given $\log 2$ as before, $\log 3 = {\cdot}4771213$, $\log 8{\cdot}1636 = {\cdot}9118818$.

If x be required value, then

$$\log x = \log (8100)^{\frac{1}{3}} + \log ({\cdot}4)^3 - \log 360 - \log ({\cdot}016)^{\frac{1}{5}}$$

$$= \frac{1}{3} \log 81 \times 100 + 3 \log \frac{4}{10} - \log 9 \times 4 \times 10 - \frac{1}{5} \log \frac{16}{1000}$$

$$= \frac{1}{3} \log 3^4 + \frac{1}{3} \log 10^2 + 3 \log 2^2 - 3 \log 10 - \log 3^2$$

$$- \log 2^2 - \log 10 - \frac{1}{5} \log 2^4 + \frac{1}{5} 10^3$$

$$= \frac{4}{3}\log 3 + \frac{2}{3} + 6\log 2 - 3 - 2\log 3 - 2\log 2 - 1$$
$$- \frac{4}{5}\log 2 + \frac{3}{5}$$
$$= \frac{16}{5}\log 2 - \frac{2}{3}\log 3 - 2\frac{11}{15}$$
$$= \cdot 963296 - \cdot 3180809 - 2\cdot 7333333$$
$$= -2\cdot 0881182$$
$$= \bar{3}\cdot 9118818.$$

Comparing this result with the last of the given logarithms,

$$x = \cdot 0081636.$$

(3.) Given $L \cos 66° 44' = 9\cdot 5966093$, tab. diff. for $1' = 2937$; find $L \cos 66° 44' 5''$.

$$\text{Difference for } 60'' = \cdot 0002937;$$
$$\therefore \text{ pro. part for } 5'' = \frac{5}{60} \text{ of } \cdot 0002937$$
$$= \cdot 0000245,$$

and $$L \cos 66° 44' = 9\cdot 5966093;$$
$$\therefore L \cos 66° 44' 5'' = 9\cdot 5965858.$$

(4.) Given $L \sin 24° 13' = 9\cdot 6129833$,
$L \sin 24° 14' = 9\cdot 6132641$;
find A, when $L \sin A = 9\cdot 6130957$.

By subtraction,

$$\text{difference for } 60'' = \cdot 0002808,$$

and $$\text{pro. part of log} = \cdot 0001124;$$
$$\therefore \text{ pro. part of angle} = \frac{1124}{2808} \text{ of } 60'' = 24'';$$
$$\therefore \text{ required angle} = 24° 13' 24''.$$

EXAMPLES.

When required, $\log 2 = \cdot 30103$, $\log 3 = \cdot 4771213$.

1. Given $\log 17256 = 4\cdot 2369401$,
$\log 17257 = 4\cdot 2369653$;
find $\log 1725674$.

2. Given $\log 7\cdot 8066 = \cdot 8924619$,
$\log 7\cdot 8067 = \cdot 8924675$;
find the number whose logarithm is $\bar{1}\cdot 8924652$.

3. Given $\log \cdot 20967 = \bar{1}\cdot 3215363$, diff. for $1 = 207$, find the number whose logarithm is $-1\cdot 6784533$.

4. Construct a table of proportional parts corresponding to a difference of 96.

5. Construct a table of proportional parts for the difference between the logarithms,

$$\log 16533 = 4\cdot 2183517,$$
$$\log 1\cdot 6534 = \cdot 2183729;$$

and find the number whose logarithm is $\bar{3}\cdot 2183736$.

6. Given $\log 2$ and $\log 3$; find $\log \sqrt[3]{\frac{1}{3}}$ and $\log (5\frac{1}{2})^{-\frac{1}{2}}$.

7. Calculate the value of $\sqrt[3]{\cdot 0000083825}$; having given

$$\log 8\cdot 3825 = \cdot 9233736, \quad \log 2\cdot 0313 = \cdot 3077741,$$
$$\log 2\cdot 0314 = \cdot 3077954.$$

8. Find $(1\cdot 05)^{20}$, having given $\log 2$, $\log 3$, $\log 7 = \cdot 84510$, $\log 2\cdot 653 = \cdot 42374$, $\log 2\cdot 654 = \cdot 42390$.

9. Given $\log 1\cdot 3107 = \cdot 1175034$,
$\log 131\cdot 08 = 2\cdot 1175364$,
and $\log 5 = \cdot 69897$;
find the seventeenth root of 131072.

10. Given $\log 3$, find $\log \left\{(2 \,.\, 7)^3 \times (\cdot 81)^{\frac{4}{3}} \div (90)^{\frac{5}{4}}\right\}$.

11. Required the value of $\dfrac{(15\cdot 6)^{\frac{1}{3}} \times (\cdot 0045)^{\frac{4}{3}}}{(\cdot 00065)^{\frac{1}{3}}}$.

Given $\log 2$, $\log 3$, $\log 13 = 1\cdot 1139434$, $\log 1\cdot 7027 = \cdot 2311381$, $\log 17028 = 4\cdot 2311636$.

12. Prove that $\log_a b \times \log_b a = 1$.

13. Prove that $a^{\log b} = b^{\log a}$.

14. Given $\log 2$, and that $\log_a x = \dfrac{\log_b x}{\log_b a}$, find the logarithm of 1000 to the base 25.

15. Describe the advantages of logarithms in conducting cumbrous arithmetical computations.

16. How arises the advantage of common logarithms over Napierian?

17. What is the use of a table of proportional parts? and how may its absence be supplied?

18. Show that if $c = (a+b) \sin \frac{C}{2} \sec \phi$, then

$$\log c = \log (a+b) + L \sin \frac{C}{2} + L \sec \phi - 20.$$

19. Prove that

$$2L \tan \frac{A}{2} = \log (s-b) + \log (s-c) - \log s - \log (s-a) + 20.$$

20. Given that $c = a \frac{\sin C}{\sin A}$, and $\sin C = \frac{c}{a} \sin A$,

show that $\log c = \log a + L \sin C - L \sin A$,

and $L \sin C = \log c + L \sin A - \log a$.

21. Given $\cos 57° 32' = \cdot 5368069$,

$\cos 57° 33' = \cdot 5365634$,

find $\cos 57° 32' 36''$.

22. Given $\sin 47° 4' = \cdot 7321467$, diff. for $1' = 1982$,
find the angle whose natural sine is $\cdot 7321564$.

23. Given $\cos 75° 56' = \cdot 2430507$, diff. for $60'' = 2822$,
find the angle whose natural cosine is $\cdot 2427193$.

24. Given $L \cos 60° 25' = 9 \cdot 6934534$,

$L \cos 60° 26' = 9 \cdot 6932308$,

find $L \cos 60° 25' 36''$.

25. Given $L \tan 12° = 9 \cdot 3274745$, diff. for $1' = 6208$,
find $L \tan 12° 0' 12''$.

26. Given $L \cot 53° 32' 40'' = 9 \cdot 8685041$, diff. for $10'' = 441$,
find $L \cot 53° 32' 42''$.

27. Given $L \sin 53° 25' = 9 \cdot 9047106$, diff. for $60'' = 938$,
find the angle whose L sine is $9 \cdot 9047423$.

28. Given $L \cot 37° 40' = 10 \cdot 1124058$, diff. for $1' = 2612$,
find A, when $L \cot A = 10 \cdot 1123463$.

CHAPTER XV.

SOLUTION OF TRIANGLES WITH LOGARITHMS.—PROBLEMS.

140. FORMULÆ have been established by means of which logarithms may be applied to all the cases of solution of triangles except one; namely that in which two sides and the included angle are given. We shall now obtain a formula for that case adapted to logarithmic computation; after which all the cases will be reconsidered for the purpose of fully exhibiting the application of logarithms to each of them.

141. *Tan* $\frac{1}{2}(A-B)$ *in terms of* C, a, b.

$$\text{Since } \frac{\sin A}{\sin B} = \frac{a}{b},$$

by theory of proportion,

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{a-b}{a+b},$$

or
$$\frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}} = \frac{a-b}{a+b};$$

therefore
$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\tan\frac{A+B}{2};$$

and
$$\tan\frac{A+B}{2} = \tan\left(90^\circ - \frac{C}{2}\right) = \cot\frac{C}{2};$$

therefore
$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}.$$

142. *Given two angles and a side, to solve the triangle.*

Let A, B, a be given; then C, b, c are required.

(i.) $C = 180° - (A+B)$.

(ii.) $\frac{b}{a} = \frac{\sin B}{\sin A}$, $\therefore\ b = a\frac{\sin B}{\sin A}$;

$\therefore\ \log b = \log a + L\sin B - L\sin A$.

(iii.) $\frac{c}{a} = \frac{\sin C}{\sin A}$, $\therefore\ c = a\frac{\sin C}{\sin A}$;

$\therefore\ \log C = \log a + L\sin C - L\sin A$,

and the triangle is solved.

If, instead of a, the given side had been b or c; since C is immediately known from $A+B$, the solution would have been exactly similar.

143. *Given two sides and the angle opposite to one of them, to solve the triangle.*

Let A, a, b be given; then B, C, c are required.

(i.) $\frac{\sin B}{\sin A} = \frac{b}{a}$, $\therefore\ \sin B = \frac{b}{a}\sin A$;

$\therefore\ L\sin B = \log b + L\sin A - \log a$.

(ii.) $C = 180° - (A+B)$.

(iii.) $c = a\frac{\sin C}{\sin A}$,

$\therefore\ \log c = \log a + L\sin C - L\sin A$.

It should be remembered that this is the ambiguous case. (See Chap. XI.)

144. *Given two sides and the included angle, to solve the triangle.*

Let a, b, C be given; then A, B, c are required.

(i.) By a formula already proved,

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2},$$

$$\therefore\ L\tan\frac{A-B}{2} = \log(a-b) + L\cot\frac{C}{2} - \log(a+b);$$

and $\frac{A-B}{2}$ is determined.

Again $$A+B=180^\circ-C,$$
$$\therefore\ \frac{A+B}{2}=90^\circ-\frac{C}{2};$$
which determines $\frac{A+B}{2}$. Now it is easily seen that
$$A=\frac{A+B}{2}+\frac{A-B}{2},$$
and $$B=\frac{A+B}{2}-\frac{A-B}{2};$$
and therefore both are determined. Then

(ii.) $$c=a\frac{\sin C}{\sin A},$$
$$\therefore\ \log c=\log a+L\sin C-L\sin A,$$
and the triangle is completely solved.

A method is usually given for finding c, without first finding A, B. This will follow the consideration of the fourth case.

145. *Given the three sides, to solve the triangle.*

Here a, b, c are given, and A, B, C required. We may find A and B successively by means of one of the formulæ
$$\sin\frac{A}{2}=\sqrt{\left\{\frac{(s-b)\ (s-c)}{bc}\right\}},\quad \cos\frac{A}{2}=\sqrt{\left\{\frac{s\,(s-a)}{bc}\right\}},$$
$$\tan\frac{A}{2}=\sqrt{\left\{\frac{(s-b)\,(s-c)}{s\,(s-a)}\right\}},\quad \cot\frac{A}{2}=\sqrt{\left\{\frac{s\,(s-a)}{(s-b)\,(s-c)}\right\}}$$
&c. &c.

all of which are adapted to logarithmic computation.

Thus,
$$L\sin\frac{A}{2}=\frac{1}{2}\{\log(s-b)+\log(s-c)-\log b-\log c\}+10.$$
Afterwards C follows from
$$C=180^\circ-(A+B).$$

It may be noticed that if but one angle is required, the first or second of the above formulæ is the best; but if more than one, then the third or fourth; in

applying which, four logarithms alone will be required; namely, those of s, $s-a$, $s-b$, $s-c$.

Again, for a reason which cannot here be entered into, when the angle is less than 45°, and especially when much less, it is best to use the first or third formula; but when greater than 45°, and especially when near 90°, it is best to use the second or fourth.

146. Angles are sometimes employed in adapting expressions to logarithmic computation, which otherwise have no connection with those expressions. Such angles are called *subsidiary angles*.

147. *Given two sides of a triangle and the included angle, to find the third side by means of a formula adapted to logarithms, without first finding the angles adjacent to it.*

Let a, b, C be the given parts of the triangle; then c is required.

$$\begin{aligned} c^2 &= a^2+b^2-2ab\cos C \\ &= (a^2+b^2)\left(\cos^2\frac{C}{2}+\sin^2\frac{C}{2}\right)-2ab\left(\cos^2\frac{C}{2}-\sin^2\frac{C}{2}\right) \\ &= (a+b)^2\sin^2\frac{C}{2}+(a-b)^2\cos^2\frac{C}{2} \\ &= (a+b)^2\sin^2\frac{C}{2}\left\{1+\frac{(a-b)^2\cos^2\frac{C}{2}}{(a+b)^2\sin^2\frac{C}{2}}\right\} \\ &= (a+b)^2\sin^2\frac{C}{2}\left\{1+\left(\frac{a-b}{a+b}\cot\frac{C}{2}\right)^2\right\}. \end{aligned}$$

Introducing a subsidiary angle, take ϕ such that

$$\tan\phi = \frac{a-b}{a+b}\cot\frac{C}{2}.$$

Since there are tangents of angles of all magnitudes, ϕ may always be found. Then

$$c^2 = (a+b)^2\sin^2\frac{C}{2}(1+\tan^2\phi),$$

$$\therefore\ c = (a+b)\sin\frac{C}{2}\sec\phi.$$

To obtain ϕ, we have

$$L \tan \phi = \log (a-b) + L \cot \frac{C}{2} - \log (a+b);$$

and then for c

$$\log c = \log (a+b) + L \sin \frac{C}{2} + L \sec \phi - 20.$$

148. The following will exemplify practically the preceding propositions:—

(1.) If in the triangle ABC, $a=1000$, $B=62° 5'$, $C=45° 41' 32''$; find b, having given

$$L \sin 62° 5' = 9{\cdot}9462702, \quad \log 9{\cdot}27929 = {\cdot}9675148,$$
$$L \sin 72° 13' = 9{\cdot}9787365, \quad \text{diff. for } 1' = 405.$$

(i.) $A = 180° - (B+C) = 180° - 170° 46' 32'' = 72° 13' 28''$.

(ii.) $b = a \dfrac{\sin B}{\sin A}$;

$$\therefore \log b = \log 1000 + L \sin 62° 5' - L \sin 72° 13' 28'';$$
$$\text{and } L \sin 72° 13' = 9{\cdot}9787365;$$

$\therefore$ adding $\frac{28}{60}$ of ·0000405, or ·0000189,

$$L \sin 72° 13' 28'' = 9{\cdot}9787554;$$
$$\therefore \log b = 3 + 9{\cdot}9462702 - 9{\cdot}9787554 = 2{\cdot}9675148;$$
$$\therefore b = 927{\cdot}929.$$

(2.) Two sides of a triangle are 500 and 400, and contain an angle of 54° 36′ 24″; find the other angles.

Given $\log 3 = {\cdot}4771213$, $L \tan 12° 9' = 9{\cdot}3330532$.

$L \cot 27° 18' = 10{\cdot}2872338$, diff. for $60'' = 3100$.

Let a, b, C be given. Then, first finding $\dfrac{A-B}{2}$,

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{100}{900} \cot \frac{C}{2} = \frac{1}{9} \cot \frac{C}{2};$$

$$\therefore L \tan \frac{A-B}{2} = L \cot 27° 18' 12'' - \log 9.$$

Now $\quad L \cot 27° 18' = 10{\cdot}2872338;$

subtract $\frac{12}{60}$ of ·0003100, or ·0000620;

then $\quad L \cot 27° 18' 12'' = 10{\cdot}2872958.$

Also $\quad \log 9 = 2 \log 3 = {\cdot}9542426;$

$$\therefore L \tan \frac{A-B}{2} = 10{\cdot}2872958 - {\cdot}9542426$$
$$= 9{\cdot}3330532;$$

$$\therefore \frac{A-B}{2} = 12^\circ\ 9',$$

and $$\frac{A+B}{2} = \frac{180^\circ - C}{2} = 62^\circ\ 41'\ 48''.$$

Adding, $$A = 74^\circ\ 50'\ 48'';$$

subtracting, $$B = 50^\circ\ 32'\ 48''.$$

(3.) The sides of a triangle are 32, 40, 66; find the greatest angle, having given

$$\log 207 = 2{\cdot}3159703, \quad \log 1073 = 3{\cdot}0305997,$$

$$L \cot 66^\circ\ 18' = 9{\cdot}6424342, \quad \text{diff. for } 1' = 3433.$$

The data here indicate which formula is to be used; namely,

$$\cot^2 \frac{C}{2} = \frac{s(s-c)}{(s-a)(s-b)}.$$

Taking a, b, c as 32, 40, 66, respectively, we have

$$2s = 32 + 40 + 66 = 138,$$

$$\therefore s = 69, \quad s-a = 37, \quad s-b = 29, \quad s-c = 3;$$

$$\therefore \cot^2 \frac{C}{2} = \frac{69 \times 3}{29 \times 37} = \frac{207}{1073};$$

$$\therefore 2L \cot \frac{C}{2} = \log 207 - \log 1073 + 20$$

$$= 2{\cdot}3159703 - 3{\cdot}0305997 + 20$$

$$= 19{\cdot}2853706;$$

$$\therefore L \cot \frac{C}{2} = 9{\cdot}6426853.$$

Subtract $$L \cot 66^\circ\ 18' = 9{\cdot}6424342;$$

then $$\text{pro. part of log} = {\cdot}0002513;$$

$$\therefore \text{pro. part of angle} = \frac{2513}{3433} \text{ of } 60'' = 44''.$$

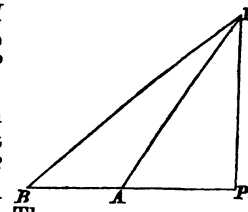
$$\therefore \frac{C}{2} = 66^\circ\ 18' - 44''$$

$$= 66^\circ\ 17'\ 16'';$$

$$\therefore C = 132^\circ\ 34'\ 32''.$$

149. The variety of useful problems in which the solution of triangles is involved is very great. The following are among those most commonly met with.

150. *To find the height and distance of an inaccessible object upon a horizontal plane.*



Let the inaccessible object PH be observed from A; and let the height PH and the distance AP be required.

From A measure a line AB in PA produced if possible; and let its length be a. At A and B observe the angles PAH and PBH; which are α, β, suppose. Then

$$AHB = PAH - PBH = \alpha - \beta,$$

and is known. And

$$AH = AB \frac{\sin PBH}{\sin AHB};$$

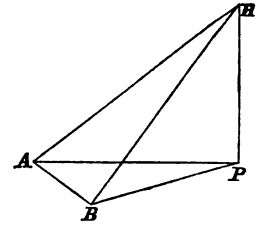
$$\therefore\ PH = AH \sin PAH = AB \frac{\sin PAH \sin PBH}{\sin AHB};$$

or
$$PH = a \frac{\sin \alpha \sin \beta}{\sin (\alpha - \beta)},$$

and $\log PH = \log a + L \sin \alpha + L \sin \beta - L \sin (\alpha - \beta) - 10.$

Similarly,
$$PA = a \frac{\cos \alpha \sin \beta}{\sin (\alpha - \beta)},$$

and $\log PA = \log a + L \cos \alpha + L \sin \beta - L \sin (\alpha - \beta) - 10;$ and therefore both are known.



If it be not possible to measure AB in AP produced; let it have *any* direction from A. Observe the angles HAP, PBA, and PAB; which are α, β, γ, suppose. Then

$$AP = AB \frac{\sin \beta}{\sin \{\pi - (\beta + \gamma)\}},$$

or
$$AP = a \frac{\sin \beta}{\sin (\beta + \gamma)},$$

and
$$\log AP = \log a + L \sin \beta - L \sin (\beta + \gamma).$$

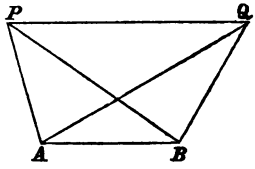
Also
$$HP = AP \tan \alpha = \frac{a \tan \alpha \sin \beta}{\sin (\beta + \gamma)},$$

and $\log HP = \log a + L \tan \alpha + L \sin \beta - L \sin (\beta + \gamma) - 10.$

151. A straight line which is measured in order that the length of it may become one of the data by means of which a required angle or distance is to be found, is called a *base line*.

AB in the preceding, and also in the following Article, is an instance of such a line.

152. *To find the distance between two inaccessible objects in a horizontal plane.*



Let P and Q be two objects which are both inaccessible, and the distance between which is required. Measure a base line AB in such a position that both P and Q are visible from either A or B; and let its length be a. Observe the angles PAB, QAB, PBA, PBQ, and let them be α, β, γ, δ, respectively.

Then, in the triangle PAB,

$$BP = a\,\frac{\sin\alpha}{\sin\{\pi-(\alpha+\gamma)\}} = \frac{a\sin\alpha}{\sin(\alpha+\gamma)};$$

and $\quad \log BP = \log a + L\sin\alpha - L\sin(\alpha+\gamma).$

Also, in the triangle QAB,

$$BQ = a\,\frac{\sin\beta}{\sin\{\pi-(\beta+\gamma+\delta)\}} = \frac{a\sin\beta}{\sin(\beta+\gamma+\delta)};$$

and $\quad \log BQ = \log a + L\sin\beta - L\sin(\beta+\gamma+\delta).$

It now remains to find PQ, having given BP, BQ, and the included angle PBQ; a method for accomplishing which has been explained.

EXAMPLES.

1. If one side of a triangle be 1000 feet, and the adjacent angles 24° 29′ 20″ and 104°; find the greatest side, given that

$$\log 12396 = 4{\cdot}093281, \text{ difference } 350;$$
$$L\sin 76^\circ = 9{\cdot}9869044;$$
$$L\sin 51^\circ\,30' = 9{\cdot}8935444, \text{ difference } 1004.$$

2. Side $a = 891{\cdot}6$, $b = 732{\cdot}8$, $A = 72° 8'$; find B.

Given $\log 8{\cdot}916 = {\cdot}9501701$, $\log 7{\cdot}328 = {\cdot}8649855$,

$L \sin 72° 8' = 9{\cdot}9785334$,

$L \sin 51° 28' = 9{\cdot}8933433$, diff. for $1' = 1007$.

3. If two sides of a plane triangle be 100 feet and 60 feet, and the included angle 42° 30′; find the other angles, given that

$\log 2 = {\cdot}3010300$, $\log \cot 21° 15' = 10{\cdot}4101858$,

$\log \tan 32° 45' = 9{\cdot}8083606$, $\log \tan 32° 44' = 9{\cdot}8080829$.

4. In a triangle ABC, in which $a = 18$, $b = 20$, $c = 22$, find $L \tan \frac{A}{2}$, having given

$\log 2 = {\cdot}3010300$, $\log 3 = {\cdot}4771213$.

5. Given $a = 18$, $b = 2$, and $C = 55°$; find A and B.

$\log 2 = {\cdot}3010300$, $L \cot 27° 30' = 10{\cdot}2835233$.

$L \tan 56° 56' = 10{\cdot}1863769$, diff. for $1' = 2763$

6. The sides of a triangle are 4, 5, 6; find B, having given $\log 2$ as before, and

$L \cos 27° 53' = 9{\cdot}9464040$, diff. for $1' = 669$.

7. If $a = 9$, $b = 7$, $C = 64° 12'$, find A and B.

Given $\log 2$ as before, and

$L \tan 11° 16' = 9{\cdot}2993216$, diff. for $60'' = 6588$.

8. The ratio of two sides of a triangle is 9 : 7, and the included angle is 47° 25′; find the other angles, having given $\log 2$,

$L \tan 66° 17' 30'' = 10{\cdot}3573942$,

$L \tan 15° 53' = 9{\cdot}4541479$, diff. for $60'' = 4797$.

9. The sides of a triangle are 7, 8, 9; determine all the angles, having given $\log 2$,

$L \tan 24° 5' 40'' = 9{\cdot}6505069$, $L \tan 24° 5' 50'' = 9{\cdot}6505634$,

$L \tan 29° 12' 20'' = 9{\cdot}7474183$, $L \tan 29° 12' 30'' = 9{\cdot}7474677$.

10. If one angle of a triangle be 60°, and the sides which contain it are as 19 to 1, find the other two angles.

Given $\log 3 = {\cdot}4771213$, $L \tan 57° 19' 11'' = 10{\cdot}1928032$.

11. Given $b = 876$ yards, $c = 113$ yards, and $A = 57° 3' 27''$; find the area of the triangle ABC.

Given $\log 4{\cdot}38 = {\cdot}6414741$, $\log 113 = 2{\cdot}0530784$,

$\log 4{\cdot}1536 = {\cdot}6184247$, diff. $= 105$.

12. The side BC of a triangle is 254 feet long, the angles B and C are 16° and 64° respectively; find the length of AC, having given

$$\log 2{\cdot}54 = 0{\cdot}4048337, \quad L \sin 80^\circ = 9{\cdot}9933515,$$
$$\log 7{\cdot}109 = 0{\cdot}8518085, \quad L \sin 16^\circ = 9{\cdot}4403381,$$
$$\log 7{\cdot}11 = 0{\cdot}8518696.$$

13. In a right-angled triangle, the lengths of the two sides are 9·65 and 12·24 yards respectively; find the angles.

Given $\log \tan 38^\circ 15' = 9{\cdot}8967116$, $\log \tan 38^\circ 16' = 9{\cdot}8969714$, $\log_{10} 2$, $\log_{10} 153 = 2{\cdot}1846914$, $\log_{10} 193 = 2{\cdot}2855573$.

14. A flag-staff, a feet high, on the top of a tower, is seen, from a certain point in the horizontal plane on which the tower stands, to subtend at the eye of the observer an equal angle (A) with the tower itself. Show that the height of the tower $= a \cos 2A$.

15. A man 6 feet high, standing on the top of a mast, subtends an angle whose tangent $= \frac{1}{10}$ at a point on the deck 33 feet from the foot of the mast; find the height of the mast.

16. From a station B at the base of a mountain, its summit A is seen at an elevation of 60°; after walking one mile towards the summit of a plane, making an angle of 30° with the horizon, to another station C, the angle BCA is observed to be 135°; find the height of the mountain.

17. A tower standing on a horizontal plane is surrounded by a moat, which is as wide as the tower is high; a person on the top of another tower whose height is a, and whose distance from the moat is c, observes that the first tower subtends an angle of 45°; find the height of the first tower.

18. The angular elevation of a tower at a place A due south of it is 30°; and at a place B, due west of A, and at the distance a from it, the elevation is 18°; show that the height of the tower is

$$\frac{a}{\sqrt{(2+2\sqrt{5})}}.$$

19. The elevation of a tower standing on a horizontal plane is observed; a feet nearer, it is found to be 45°; b feet nearer still, it is found to be the complement of what it was at the first station; show that the height of the tower is $\dfrac{ab}{a-b}$ feet.

20. At each of three stations in the same horizontal plane, and at given distances from each other, the elevation of a tower is the same angle. Show how to find the height and distance of it.

21. The upper half of a post seen from a point on a level with the foot of the post subtends an angle whose tangent $= \frac{1}{3}$; find the angle subtended by the whole post.

22. A column on a pedestal 20 feet high subtends an angle of 45°

to a person on the ground; on approaching 20 feet, it again subtends an angle of 45°. Find the height of the column.

23. From C, the foot of a tower, AC is drawn in the horizontal plane; the angle of elevation of B, the top of the tower, as seen from A, is α; AD ($= a$ feet) is measured from A, in the horizontal plane perpendicular to AC; at D the angle of elevation of the tower is β. Show that the height of the tower is

$$\frac{a \sin \alpha \sin \beta}{\sqrt{\sin (\alpha+\beta) \sin (\alpha-\beta)}}.$$

24. On the bank of a river, there is a column 200 feet high supporting a statue 30 feet high; the statue to an observer on the opposite bank subtends an equal angle with a man 6 feet high standing at the base of the column; required the breadth of the river.

25. The elevation of a tower on a horizontal plane is observed; on advancing a feet nearer, its elevation is found to be the complement of the former. On advancing again, its elevation is found to be double of its first elevation: show that the last station is $\frac{a}{2}$ feet from the foot of the tower.

26. The top of a pole, placed against a wall at an angle A with the horizontal, just touches the coping, and when its foot is moved a feet further from the wall, and its angle of inclination is B, it rests on the sill of a window; prove that the perpendicular distance between the coping and the sill $= a \cot \frac{A+B}{2}$.

27. A ship which is known to be sailing due East at 12 miles an hour, was observed at noon to be 15° to the East of South; at 1.30 p.m. it was seen in the South-East; what was its distance when first seen?

28. A person wishing to know the distance of a tower C, measures a line AB 100 yards long, and observes that the angles BAC and ABC are respectively 53° 20′ and 59° 30′; how far is C from A?

$$L \sin 59° 30' = 9{\cdot}9353204, \quad L \cos 22° 50' = 9{\cdot}9645602,$$
$$\log 93489 = 4{\cdot}9707605.$$

29. A tower 150 feet high throws a shadow 75 feet long upon the horizontal upon which it stands. Find the sun's altitude, having given

$$\log 2 = {\cdot}3010300, \quad L \tan 63° 26' = 10{\cdot}3009994,$$
$$L \tan 63° 27' = 10{\cdot}3013153.$$

30. A person, standing on the bank of a river, observes the elevation of a tree on the opposite bank to be 51°, and when he

retires 30 feet from the river's edge, he finds the elevation to be 46°; find the breadth of the river, having given

$$\log 1{\cdot}558 = {\cdot}19263, \quad L \sin 46^\circ = 9{\cdot}85693,$$
$$\log 3 = {\cdot}47712, \quad L \sin 39^\circ = 9{\cdot}79887,$$
$$L \sin 5^\circ = 8{\cdot}94029.$$

31. A person observes the angle of elevation of a mountain to be 30°, and approaching 600 yards nearer, to be 60°; find its height.

Given $\log 3 = {\cdot}47712$, $\log 51 = 1{\cdot}70757$, $\log 52 = 1{\cdot}71600$.

32. A triangular field has its sides respectively 500, 600, 700 links in length; find its area, having given

$$\log_{10} 2 = {\cdot}30103, \quad \log_{10} 1{\cdot}469 = 0{\cdot}16702,$$
$$\log_{10} 3 = {\cdot}47712, \quad \log_{10} 1{\cdot}470 = 0{\cdot}16732.$$

33. ABC is a triangular piece of ground; BC is 21 chains, AC is $10\frac{1}{2}$ chains, and angle $ACB = 36^\circ\ 52'\ 12''$; find the remaining angles.

Given $\log_{10} 2 = {\cdot}3010300$, $\log_{10} 15 = 1{\cdot}1760913$,
$L \cot 18^\circ\ 26'\ 6'' = 10{\cdot}4771213$.

34. $ABCD$ is a rectangular piece of water, the dimensions of which are required; but, on account of the nature of the ground, the only measures which can be taken are the angles which BC subtends at A, and at a point P, 220 feet from A in BA produced, the former angle being 71°, and the latter 55°. Find the length and breadth of the rectangle, having given

$$\log 2{\cdot}2 = {\cdot}34242, \quad L \sin 16^\circ = 9{\cdot}44034,$$
$$\log 2{\cdot}1285 = {\cdot}32808, \quad L \sin 55^\circ = 9{\cdot}91336,$$
$$\log 6{\cdot}1817 = {\cdot}79111, \quad L \sin 71^\circ = 9{\cdot}97567,$$
$$L \cos 71^\circ = 9{\cdot}51264.$$

35. In a survey it is found necessary to continue a straight line AB past an obstacle which, from its height, hinders the view of the parts beyond. A line BD is therefore measured at right angles to AB, and from the point D, lines DP, DQ are drawn which clear the obstacle; the angles BDP and BDQ are found to contain 41° and 68° respectively, the distance BD being 180 yards. What lengths must be set off along DP and DQ to ensure that PQ shall be in the prolongation of AB?

Given $\log_{10} 1{\cdot}8 = 0{\cdot}255273$, $L \cos 41^\circ = 9{\cdot}877780$,
$\log_{10} 2{\cdot}385 = 0{\cdot}377488$, $L \sin 22^\circ = 9{\cdot}573575$,
$\log_{10} 4{\cdot}805 = 0{\cdot}681693$.

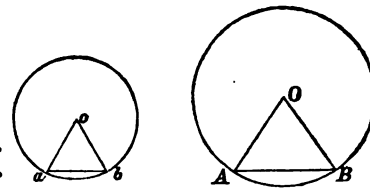
36. Two hills rise at the same point, with inclination of 60° and 40° to the horizon. At a distance of 64 feet from the base of the lower hill, the angles of elevation of the bottom and top of a vertical object on the other hill are 40° and 70°. Find the height of the object, having given

$$L \tan 20^\circ = 9{\cdot}5610695, \quad L \cos 40^\circ = 9{\cdot}8842540,$$
$$\log 2 = {\cdot}3010300, \quad \log 29640031 = 7{\cdot}4303981.$$

CHAPTER XVI.

CIRCULAR MEASURE—INSCRIBED AND CIRCUMSCRIBED POLYGONS.

153. *Circumferences of circles vary as their radii.*



Take any two circles whose circumferences are C, c, and radii R, r; and let their centres be O, o respectively. Let AB and ab be sides of regular polygons of n sides inscribed in these circles. Join OA, OB, oa, ob.

It is evident that OAB and oab are similar triangles; and therefore that

$$\frac{AB}{ab} = \frac{OA}{oa} = \frac{R}{r};$$

wherefore
$$\frac{n \,.\, AB}{n \,.\, ab} = \frac{R}{r};$$

that is, the perimeters of the polygons have to one another the same ratio as the radii of the circles in which they are respectively inscribed.

Now, assuming that, when n is indefinitely great, the perimeters of the polygons coincide with the circumferences of the circles in which they are inscribed, since the above equation is true for all values of n, we have

$$\frac{C}{c} = \frac{R}{r}.$$

Since also $\frac{C}{R} = \frac{c}{r}$, we express the same proposition by saying that the ratio of the circumference to the radius is the same for all circles.

154. *The angle subtended at the centre of a circle by an arc equal in length to the radius is constant.*

Let ω be the angle subtended at the centre of any circle by an arc equal in length to the radius. Then, since, by Euc. VI. 33, the angles at the centre are to one another as the arcs upon which they stand,

$$\frac{\omega}{4 \text{ right angles}} = \frac{\text{are subtending } \omega}{\text{circumference}}$$

$$= \frac{r}{2\pi r} = \frac{1}{2\pi};$$

$$\therefore \omega = \frac{4 \text{ right angles}}{2\pi}.$$

Hence ω is constant and equal to $\frac{2 \text{ right angles}}{3{\cdot}14159}$, the value of which has been found in Chapter I.

It is therefore such an angle as may be adopted for the unit of angular measurement. It is also a *convenient* unit for that purpose, as will be shown next.

155. *Any angle at the centre of a circle is expressed in terms of the unit of circular measure by the ratio of the arc which subtends it to the radius of the circle.*

Let the angle ω, equal to the unit of circular measure, and α, any other angle, be both at the centre of the same circle, and let a be subtended by an arc equal to a.

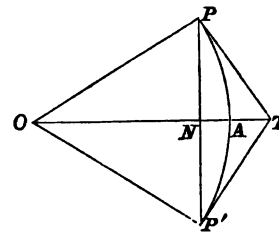
Then $$\frac{\alpha}{\omega} = \frac{a}{r}.$$

But ω is unity, therefore

$$\alpha = \frac{a}{r}.$$

Hence the ratio of subtending arc to radius expresses an angle in circular measure.

156. *The circular measure of an angle is greater than its sine, but less than its tangent.*



Let AOP be any angle whose circular measure is θ. Make AOP' also equal to θ; and from centre O describe the arc PAP'. Join PP', and draw PT, $P'T$ tangents to the circle PAP', meeting in T.

We may assume as manifest that PAP' is greater than PP' but less than PTP', and therefore that AP is greater than PN but less than PT.

Therefore, since $OA = OP$,

$$\frac{AP}{OA} \text{ is greater than } \frac{PN}{OP} \text{ but less than } \frac{PT}{OP};$$

$\therefore$ θ is greater than $\sin\theta$, but less than $\tan\theta$; which may also be expressed by saying that $\sin\theta$, θ, $\tan\theta$, are in ascending order of magnitude.

157. *The limit of the ratio* $\dfrac{\sin\theta}{\theta}$ *and also of* $\dfrac{\tan\theta}{\theta}$, *when* θ *is indefinitely diminished, is unity.*

It has been proved that θ is between $\sin\theta$ and $\tan\theta$ in value; therefore, dividing each of the three magnitudes by $\sin\theta$,

$$\frac{\theta}{\sin\theta} \text{ is between 1 and } \frac{1}{\cos\theta}.$$

Now when θ is zero, $\dfrac{1}{\cos\theta}$ is equal to 1. It follows in that case that $\dfrac{\theta}{\sin\theta}$ is 1; and therefore also $\dfrac{\sin\theta}{\theta}$ is 1.

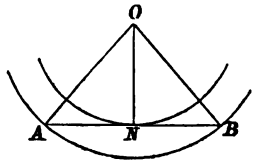
Hence, as θ is indefinitely diminished, $\dfrac{\sin\theta}{\theta}$ approaches unity in its value, which therefore is its limit; or, as it is usual to express it,

$$\frac{\sin\theta}{\theta} = 1, \text{ when } \theta = 0.$$

Therefore also, since $\dfrac{\tan\theta}{\theta} = \dfrac{\sin\theta}{\theta} \times \dfrac{1}{\cos\theta}$,

$$\frac{\tan\theta}{\theta} = 1, \text{ when } \theta = 0.$$

158. *Perimeter and area of a regular polygon in terms of the radius of the inscribed and circumscribed circle.*



Let AB be the side of a regular polygon of n sides, and O the centre of the circles inscribed and circumscribed about the polygon; also let ON and OA be the respective radii of those circles. Then ON is perpendicular to AB, and bisects the angle AOB.

Suppose $AB = a$, $ON = r$, $OA = R$.

Then since $AOB = \dfrac{2\pi}{n}$, $\therefore AON = \dfrac{\pi}{n}$;

therefore $$AB = 2AN = 2ON \tan\frac{\pi}{n},$$

or $$a = 2r \tan\frac{\pi}{n};$$

$$\therefore na = 2nr \tan\frac{\pi}{n}.$$

Also $$AB = 2AN = 2 . OA \sin\frac{\pi}{n},$$

or $$a = 2R \sin\frac{\pi}{n}, \quad \therefore na = 2nR \sin\frac{\pi}{n};$$

hence *perimeter* $= 2nr \tan\dfrac{\pi}{n}$, or $2nR \sin\dfrac{\pi}{n}$.

Again, area $AOB = ON . AN = ON . ON \tan\dfrac{\pi}{n}$,

or $$\Delta AOB = r^2 \tan\frac{\pi}{n};$$

$$\therefore n\Delta AOB = nr^2 \tan\frac{\pi}{n}.$$

Also $\Delta AOB = AN \,.\, ON = OA \sin\frac{\pi}{n} OA \cos\frac{\pi}{n}$

$$= \frac{1}{2} n\, OA^2 \sin\frac{2\pi}{n},$$

or $$\Delta AOB = \frac{1}{2} R^2 \sin\frac{2\pi}{n};$$

$$\therefore\ n\Delta AOB = \frac{n}{2} R^2 \sin\frac{2\pi}{n}.$$

Hence $$area = nr^2 \tan\frac{\pi}{n}, \text{ or } \frac{n}{2} R^2 \sin\frac{2\pi}{n}.$$

159. *Area of a regular polygon in terms of its side.*

Since $$\text{area } AOB = AN \,.\, ON = AN \,.\, AN \cot\frac{\pi}{n}$$

$$= \frac{a}{2} \times \frac{a}{2} \cot\frac{\pi}{n};$$

therefore $$n\Delta AOB = n\frac{a^2}{4} \cot\frac{\pi}{n},$$

or $$\text{area of polygon} = \frac{na^2}{4} \cot\frac{\pi}{n}.$$

160. *Circumference and area of a circle in terms of its radius.*

Suppose a polygon of n sides to be circumscribed about the circle. It may be considered as manifest that, when n is indefinitely increased, the sides of the polygon are indefinitely diminished, and the perimeter and area of the polygon approximate in magnitude to the circumference and area of the circle respectively.

Now, perimeter of polygon

$$= 2nr \tan\frac{\pi}{n} = 2\pi r \frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}};$$

and when n is indefinitely increased, $\frac{\pi}{n}$ is indefinitely

diminished, and therefore

$$\frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}} = 1;$$

therefore circumference of circle $= 2\pi r$.

Also, area of polygon

$$= nr^2 \tan\frac{\pi}{n} = \pi r^2 \frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}};$$

therefore $\qquad$ area of circle $= \pi r^2$.

161. *Area of sector of a circle.*

Let θ be the circular measure of the angle contained by the two radii which bound the sector. Then evidently

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\theta}{2\pi};$$

therefore $\qquad$ area of sector $=$ area of circle $\times \dfrac{\theta}{2\pi}$

$$= \pi r^2 \times \frac{\theta}{2\pi},$$

or $\qquad$ area of sector $= \frac{1}{2}\theta r^2$.

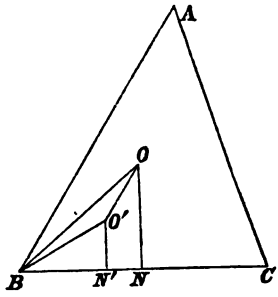
162. *Distance between the centres of circumscribed and inscribed circles of a triangle in terms of the radii of the circles.*

R and r being the respective radii of the circles, it is easy to prove, and will therefore be assumed, that

$$r = 4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}.$$

Let O be the centre of the circumscribed, and O' the

centre of the inscribed circle. Draw ON, $O'N'$ perpendiculars to BC.



Then $OB = R$, $O'N' = r$,

$$OBC = 90° - A,$$

because $BON = A$, being half of the angle $2A$ at the centre of the circumscribed circle;

and $$O'B = r \operatorname{cosec} \frac{B}{2}.$$

Then $$OO'^2 = OB^2 + O'B^2 - 2 \,.\, OB \,.\, O'B \cos OBO'$$

$$= R^2 + r^2 \operatorname{cosec}^2 \frac{B}{2} - 2Rr \operatorname{cosec} \frac{B}{2} \cos OBO',$$

and $$OBO' = \frac{1}{2} B - (90° - A) = \frac{a-c}{2};$$

therefore $OO'^2 =$

$$R^2 - 2Rr \left\{ \frac{\cos \frac{A}{2} \cos \frac{C}{2} + \sin \frac{A}{2} \sin \frac{C}{2}}{\sin \frac{B}{2}} - \frac{r \operatorname{cosec}^2 \frac{B}{2}}{2R} \right\}$$

Substitute the value which we assumed for r within the bracket, and we obtain OO'^2

$$= R^2 - 2Rr \left\{ \frac{\cos \frac{A}{2} \cos \frac{C}{2} + \sin \frac{A}{2} \sin \frac{C}{2} - 2 \sin \frac{A}{2} \sin \frac{C}{2}}{\sin \frac{B}{2}} \right\}$$

$$= R^2 - 2Rr \frac{\cos \frac{A+C}{2}}{\cos \frac{A+C}{2}}$$

$$= R^2 - 2Rr;$$

$$\therefore \quad OO' = \sqrt{R^2 - 2Rr}.$$

EXAMPLES.

1. If a be the side of a regular polygon of n sides, and R, r the radii of its circumscribed and inscribed circles, show that

$$R+r = \tfrac{1}{2}a \cot \frac{\pi}{2n}.$$

2. If the side of a pentagon inscribed in a circle be 1, the radius $= \dfrac{\sqrt{(5+\sqrt{5})}}{10}$.

3. The area of a regular hexagon inscribed in a circle is a mean proportional between the areas of an inscribed and a circumscribed equilateral triangle.

4. The number of sides of one regular polygon exceeds that of another by 1, and an angle of the first exceeds that of the second by $4°$; find the number of sides in each.

5. If r, r' be the radii of the escribed circles touching externally the sides of a right-angled triangle which are adjacent to the right angle, the area of the triangle is equal to rr'.

6. An arc of $120°$ in one circle is equal to the whole circumference of another. Compare their radii, and find how many degrees of the circumference of the smaller circle are equal in length to the radius of the larger.

7. The square of the side of a pentagon inscribed in a circle is equal to the sum of the squares of the sides of a regular hexagon and decagon inscribed in the same circle.

8. The diameter of a circle is divided in the ratio of $1\frac{2}{3} : 2\frac{1}{2}$, and circles are described on the segments as diameters. Compare the areas of the three circles.

9. If r be the radius of the inscribed circle of a triangle, show that the product of the three perpendiculars from the angles on the sides will be $\dfrac{(a+b+c)^3}{abc} r^3$.

10. The area of a regular polygon inscribed in a circle : that of the corresponding circumscribed polygon $= 3 : 4$; find the number of sides.

11. The sides of a triangle are in arithmetical progression, and the distance between the centres of the inscribed and circumscribed circles is a mean proportional between the greatest and least; show that the sides are as $\sqrt{5}-1$, $\sqrt{5}$, $\sqrt{5}+1$.

12. If α, β, γ be the distances of the centre of the inscribed circle of a triangle from the angles, and r_a, r_b, r_c be the radii of the escribed circles, prove that if $\dfrac{2}{\beta^2} = \dfrac{1}{\alpha^2} + \dfrac{1}{\gamma^2}$, then r_a, r_b, r_c are in Arithmetical Progression.

13. Prove that, if the radius of the circumscribing circle of a triangle be equal to the perpendicular drawn from one of the angles on the opposite side, the product of the sines of the angles adjacent to that side is $\frac{1}{2}$.

14. Three equal circles touch each other; show that the space between them is nearly equal to the square described upon a fifth part of the diameter; find the area when the circles are unequal and the radii as the numbers 1, 2, 3.

15. If the side of a regular inscribed polygon be an mth part of the radius, show that the secant of the angle subtended by it at the centre will be $\frac{2m^2}{2m^2-1}$.

16. If p_1, p_2, p_3 be the perpendiculars from the angles of a plane triangle upon the opposite sides, r the radius of the inscribed circle, and r_a, r_b, r_c those of the escribed circles, prove that

$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=\frac{1}{r}=\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}.$$

17. The centres of the three escribed circles of a triangle are joined; show that the area of the triangle thus formed is $\frac{abc}{2r}$, where r is the radius of the inscribed circle of the original triangle.

18. If the alternate angles of a regular hexagon be joined so as to form another regular hexagon, and again the alternate angles of the latter hexagon be joined, and so on, show that the sum of the areas of all the figures so formed $=\frac{A}{2}$, where A is the area of the original figure. And generally, if the figure has n sides, the sum

$$=\frac{A\cos^2\frac{2\pi}{n}}{\sin\frac{3\pi}{n}\sin\frac{\pi}{n}}.$$

Explain the cases where $n=3$ or 4.

19. If an equilateral triangle be described with its angular points on the sides of a given right-angled isosceles triangle; and one side parallel to the hypotenuse, its area will be

$$2a^2\sin 60^\circ(\sin 15^\circ)^2,$$

where a is a side of the given triangle.

20. If R be the radius of the circle circumscribing a triangle, ρ the radius of an escribed circle; the distance of the centres of these circles is $\sqrt{(R^2+2R\rho)}$.

CHAPTER XVII.

DE MOIVRE'S THEOREM AND APPLICATIONS OF IT.

163. *De Moivre's Theorem.—For the values of n,*

$$(\cos\theta+\sqrt{-1}\sin\theta)^n=\cos n\theta+\sqrt{-1}\sin n\theta.$$

By multiplication,

$$(\cos\alpha+\sqrt{-1}\sin\alpha)(\cos\beta+\sqrt{-1}\sin\beta)$$
$$=\cos\alpha\cos\beta-\sin\alpha\sin\beta+\sqrt{-1}(\sin\alpha\cos\beta+\cos\alpha\sin\beta)$$
$$=\cos(\alpha+\beta)+\sqrt{-1}\sin(\alpha+\beta).$$

Similarly it will be found that

$$\{\cos(\alpha+\beta)+\sqrt{-1}\sin(\alpha+\beta)\}(\cos\gamma+\sqrt{-1}\sin\gamma)$$
$$=\cos(\alpha+\beta+\gamma)+\sqrt{-1}\sin(\alpha+\beta+\gamma);$$

that is,

$$(\cos\alpha+\sqrt{-1}\sin\alpha)(\cos\beta+\sqrt{-1}\sin\beta)(\cos\gamma+\sqrt{-1}\sin\gamma)$$
$$=\cos(\alpha+\beta+\gamma)+\sqrt{-1}\sin(\alpha+\beta+\gamma).$$

Let this process be continued to n factors, and then suppose each factor equal to $\cos\theta+\sqrt{-1}\sin\theta$; then

$$(\cos\theta+\sqrt{-1}\sin\theta)^n=\cos n\theta+\sqrt{-1}\sin n\theta;$$

which proves the theorem when n is integral.

Next, let n be negative and equal to $-m$ suppose; then

$$(\cos\theta+\sqrt{-1}\sin\theta)^{-m}=\left(\frac{1}{\cos\theta+\sqrt{-1}\sin\theta}\right)^m$$
$$=\left(\frac{\cos^2\theta+\sin^2\theta}{\cos\theta+\sqrt{-1}\sin\theta}\right)^m$$
$$=(\cos\theta-\sqrt{-1}\sin\theta)^m$$
$$=\cos m\theta-\sqrt{-1}\sin m\theta$$
$$=\cos(-m\theta)+\sqrt{-1}\sin(-m\theta)$$

$\therefore\ (\cos\theta+\sqrt{-1}\sin\theta)^n=\cos n\theta+\sqrt{-1}\sin n\theta,$ when n is negative.

Lastly, let n be fractional, and equal to $\frac{p}{q}$; then

$$(\cos\theta+\sqrt{-1}\sin\theta)^p=\cos p\theta+\sqrt{-1}\sin p\theta$$
$$=\left(\cos\frac{p}{q}\theta+\sqrt{-1}\sin\frac{p}{q}\theta\right)^q.$$

Therefore, extracting roots, one value of

$$(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{p}{q}}=\cos\frac{p}{q}\theta+\sqrt{-1}\sin\frac{p}{q}\theta;$$

and the theorem is true for fractional indices.

The theorem is therefore true for positive or negative, integral or fractional indices.

164. It remains to notice that the values of the expression

$$(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{p}{q}}$$

are q in number, of which only one is given above.

Since the expression for the angles which have both the same sine and cosine as θ, is $2n\pi+\theta$, where n is any integer, we may write for the root already obtained

$$\cos\frac{p}{q}(2n\pi+\theta)+\sqrt{-1}\sin\frac{p}{q}(2n\pi+\theta).$$

By giving successively the values 1, 2, 3, ...q to n in this expression, we obtain *all* the values of the root.

165. *Functions of the multiple angle in terms of those of the simple angle.*

Since

$$\cos n\theta+\sqrt{-1}\sin n\theta=(\cos\theta+\sqrt{-1}\sin\theta)^n,$$

expanding by the Binomial Theorem,

$$\cos n\theta + \sqrt{-1}\sin n\theta = \cos^n\theta + n\sqrt{-1}\cos^{n-1}\theta\sin\theta$$
$$-\frac{n(n-1)}{1.2}\cos^{n-2}\theta\sin^2\theta$$
$$-\frac{n(n-1)(n-2)}{1.2.3}\sqrt{-1}\cos^{n-3}\theta\sin^3\theta + \ldots\ldots$$

Therefore, equating real terms on opposite sides of this equation,

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{1.2}\cos^{n-2}\theta\sin^2\theta$$
$$+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\cos^{n-4}\theta\sin^4\theta - \ldots\ldots$$

Equating the unreal terms, and then dividing both sides by $\sqrt{-1}$,

$$\sin n\theta = n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{1.2.3}\cos^{n-3}\cos^3\theta$$
$$+\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}\cos^{n-5}\cos^5\theta - \ldots$$

From the above results,

$$\tan n\theta =$$
$$\frac{n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{1.2.3}\cos^{n-3}\theta\sin^3\theta + \ldots.}{\cos^n\theta - \frac{n(n-1)}{1.2}\cos^{n-2}\theta\sin^2\theta + \ldots.};$$

therefore, dividing numerator and denominator of the fraction by $\cos^n\theta$,

$$\tan n\theta =$$
$$\frac{n\tan\theta - \frac{n(n-1)(n-2)}{1.2.3}\tan^3\theta + \ldots..}{1 - \frac{n(n-1)}{1.2}\tan^2\theta + \frac{n(n-1)(n-2)(n-3)}{1.2.3,4}\tan^4\theta - \ldots}.$$

166. Space does not allow of an investigation of the general terms of these expansions; which must therefore be left as an instructive exercise to the student. The same remark will apply in other similar cases in this chapter.

167. *Express the tangent of the sum of any angles in terms of the tangents of the angles.*

Since

$$\cos\alpha\,(1+\sqrt{-1}\tan\alpha) = \cos\alpha + \sqrt{-1}\sin\alpha$$
$$\cos\beta\,(1+\sqrt{-1}\tan\beta) = \cos\beta + \sqrt{-1}\sin\beta$$
$$\cos\gamma\,(1+\sqrt{-1}\tan\gamma) = \cos\gamma + \sqrt{-1}\sin\gamma$$
$$\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$$

therefore multiplying, and adopting the expression for the product of the right-hand members of these equations which has been already obtained (Art. 163), we have

$$\cos\alpha\,.\cos\beta\,.\cos\gamma\ldots\,(1+\sqrt{-1}\tan\alpha)(1+\sqrt{-1}\tan\beta)(1+\sqrt{-1}\tan\gamma)\ldots$$
$$= \cos(\alpha+\beta+\gamma+\ldots) + \sqrt{-1}\sin(\alpha+\beta+\gamma+\ldots).$$

Write s_1 for $\tan\alpha+\tan\beta+\tan\gamma+\ldots$; s_2 for the sum of the products of the tangents taken two at a time; s_3 for the sum of the products of the tangents taken three at a time; and so on. Then, by multiplying together the factors $1+\sqrt{-1}\tan\alpha$, $1+\sqrt{-1}\tan\beta$, $1+\sqrt{-1}\tan\gamma$, &c., and equating real and unreal terms, we obtain

$$\cos(\alpha+\beta+\gamma+\ldots) = \cos\alpha\cos\beta\cos\gamma\ldots(1-s_2+s_4-s_6+\ldots),$$
$$\sin(\alpha+\beta+\gamma+\ldots) = \cos\alpha\cos\beta\cos\gamma\ldots(s_1-s_3+s_5-s_7+\ldots).$$

Dividing, we have

$$\tan(\alpha+\beta+\gamma+\ldots) = \frac{s_1-s_3+s_5-s_7+\ldots}{1-s_2+s_4-s_6+\ldots}.$$

168. *To express sin α and cos α in terms of α.*

It will be seen from a former result that

$$\cos n\alpha = \cos^n\alpha\left\{1-\frac{n(n-1)}{1\,.\,2}\tan^2\alpha + \frac{n(n-1)(n-2)(n-3)}{\underline{|4}}\tan^4\alpha-\ldots\ldots\right\},$$

and

$$\sin na = \cos^n a \left\{ n \tan a - \frac{n(n-1)(n-2)}{1.2.3} \tan^3 a + \ldots \right\}.$$

Let $na = \theta$, $\therefore\ n = \dfrac{\theta}{a}$;

$$\therefore\quad \cos\theta = \cos^n a \left\{ 1 - \frac{\theta(\theta-a)}{1.2}\left(\frac{\tan a}{a}\right)^2 + \frac{\theta(\theta-a)(\theta-2a)(\theta-3a)}{1.2.3.4}\left(\frac{\tan a}{a}\right)^4 - \ldots \right\},$$

and $\sin\theta =$

$$\cos^n a \left\{ \theta \cdot \frac{\tan a}{a} - \frac{\theta(\theta-a)(\theta-2a)}{1.2.3}\left(\frac{\tan a}{a}\right)^3 + \ldots \right\}.$$

Suppose θ to remain finite, while n increases indefinitely; then a must decrease indefinitely; in which case $\cos^n a = 1$, and $\dfrac{\tan a}{a} = 1$;

$$\therefore\quad \cos\theta = 1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{\underline{|4}} - \frac{\theta^6}{\underline{|6}} + \ldots;$$

$$\sin\theta = \theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \ldots$$

These series are called "Euler's series."

169. When x denotes the expression $\cos\theta + \sqrt{-1}\sin\theta$, the values of $\dfrac{1}{x}$, $x + \dfrac{1}{x}$, $x - \dfrac{1}{x}$, $x_n + \dfrac{1}{x^n}$, and $x^n - \dfrac{1}{x^n}$ are worthy of attention.

$$\frac{1}{x} = \frac{1}{\cos\theta + \sqrt{-1}\sin\theta} = \frac{\cos^2\theta - (\sqrt{-1}\sin\theta)^2}{\cos\theta + \sqrt{-1}\sin\theta};$$

$$\therefore\quad \frac{1}{x} = \cos\theta - \sqrt{-1}\sin\theta,$$

and

$$x = \cos\theta + \sqrt{-1}\sin\theta;$$

$$\therefore\quad x + \frac{1}{x} = 2\cos\theta,$$

and

$$x - \frac{1}{x} = 2\sqrt{-1}\sin\theta.$$

Again, $x^n = (\cos\theta + \sqrt{-1}\sin\theta)^n = \cos n\theta + \sqrt{-1}\sin n\theta$,

and $\dfrac{1}{x^n} = \dfrac{1}{\cos n\theta + \sqrt{-1}\sin n\theta} = \cos n\theta - \sqrt{-1}\sin n\theta$.

Hence $$x^n + \frac{1}{x^n} = 2\cos n\theta,$$

and $$x^n - \frac{1}{x^n} = 2\sqrt{-1}\sin n\theta.$$

170. *To express $\cos^n\theta$ in terms of cosines of multiples of θ, when n is a positive integer.*

Writing x for $\cos\theta + \sqrt{-1}\sin\theta$, we have

$$2\cos\theta = x + \frac{1}{x};$$

$$\therefore\quad 2^n\cos^n\theta = \left(x + \frac{1}{x}\right)^n$$

$$= x^n + nx^{n-1}\frac{1}{x} + \frac{n(n-1)}{1.2}x^{n-2}.\frac{1}{x^2} + \ldots\ldots$$

$$\ldots\ldots + \frac{n(n-1)}{1.2}x^2\frac{1}{x^{n-2}} + nx.\frac{1}{x^{n-1}} + \frac{1}{x^n}.$$

$$= x^n + nx^{n-2} + \frac{n(n-1)}{1.2}x^{n-4} + \ldots\ldots$$

$$\ldots\ldots + \frac{n(n-1)}{1.2}\frac{1}{x^{n-4}} + n\frac{1}{x^{n-2}} + \frac{1}{x^n}$$

$$= x^n + \frac{1}{x^n} + n\left(x^{n-2} + \frac{1}{x^{n-2}}\right)$$

$$+ \frac{n(n-1)}{1.2}\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \ldots\ldots$$

Therefore, since

$$x^n + \frac{1}{x^n} = 2\cos n\theta, \quad x^{n-2} + \frac{1}{x^{n-2}} = 2\cos(n-2)\theta, \text{ \&c.};$$

substituting, and dividing both sides by 2,

$$2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{1.2}\cos(n-4)\theta$$

$$+ \ldots\ldots + \frac{n(n-1)\ldots(n-r+1)}{\underline{|r}}\cos(n-2r)\theta + \ldots\ldots$$

169. *To express $\sin^n \theta$ in terms of sines or cosines of multiples of θ, according as n is odd or even.*

Let $x = \cos\theta + \sqrt{-1}\sin\theta$, then $2\sqrt{-1}\sin\theta = x - \frac{1}{x}$;

and since $(\sqrt{-1})^n = (-1)^{\frac{n}{2}}$, therefore

(1.) *When n is odd,*

$$2^n(-1)^{\frac{n}{2}}\sin^n\theta = \left(x - \frac{1}{x}\right)^n$$

$$= x^n - nx^{n-2} + \frac{n(n-1)}{1.2}x^{n-4} - \ldots\ldots$$

$$\ldots\ldots - \frac{n(n-1)}{1.2}\frac{1}{x^{n-4}} + n\frac{1}{x^{n-2}} - \frac{1}{x^n}$$

$$= x^n - \frac{1}{x^n} - n\left(x^{n-2} - \frac{1}{x^{n-2}}\right)$$

$$+ \frac{n(n-1)}{1.2}\left(x^{n-4} - \frac{1}{x^{n-4}}\right) - \ldots\ldots$$

Therefore, since $x^n - \frac{1}{x^n} = 2\sqrt{-1}\sin n\theta$, &c., substituting and dividing by $2\sqrt{-1}$,

$$2^{n-1}(-1)^{\frac{n-1}{2}}\sin^n\theta$$

$$= \sin n\theta - n\sin(n-2)\theta + \frac{n(n-1)}{1.2}\sin(n-4)\theta - \ldots$$

$$+ (-1)^r\frac{n(n-1)\ldots\ldots(n-r+1)}{\underline{|r}}\sin(n-2r)\theta\ldots$$

(2.) *When n is even,*

$$2^n(-1)^{\frac{n}{2}}\sin^n\theta = x^n - nx^{n-2} + \frac{n(n-1)}{1.2}x^{n-4} - \ldots\ldots$$

$$+ \frac{n(n-1)}{1.2}\frac{1}{x^{n-4}} - n.\frac{1}{x^{n-2}} + \frac{1}{x^n}$$

$$= x^n + \frac{1}{x^n} - n\left(x^{n-2} + \frac{1}{x^{n-2}}\right)$$

$$+ \frac{n(n-1)}{1.2}\left(x^{n-4} + \frac{1}{x^{n-4}}\right) - \ldots\ldots$$

Therefore, substituting for $x^n + \frac{1}{x^n}$, &c., and dividing by 2,

$$2^{n-1}(-1)^{\frac{n}{2}} \sin^n \theta$$
$$= \cos n\theta - n \cos (n-2)\theta + \frac{n(n-1)}{1.2} \cos (n-4)\theta - \ldots$$
$$+ (-1)^r \frac{n(n-1)\ldots\ldots(n-r+1)}{\underline{|r}} \cos (n-2r)\theta \ldots$$

EXAMPLES.

1. Obtain one value of $\sqrt[3]{(\cos 4a + \sqrt{-1} \sin 4a)}$.
2. Find the values of $\sqrt[3]{1}$.
3. Find the values of $(-1)^{\frac{1}{3}}$.
4. Expand, in terms of the cosines of multiples of θ,
$\cos^6 \theta$ and $\sin^6 \theta$.
5. Expand, in terms of the sines of multiples of θ,
$\sin^5 \theta$ and $\sin^7 \theta$.
6. Expand, in terms of the powers of $\sin \theta$ and $\cos \theta$,
$\sin 4\theta$ and $\cos 7\theta$.
7. Expand $(\sin \theta)^{4n+2}$ in terms of cosines of multiples of θ.
8. Given $\sin\left(\frac{\pi}{6} + \theta\right) = \cdot 51$, find approximately the value of θ, neglecting powers of θ above the second.
9. Find the values of $(1)^{\frac{1}{5}}$.
10. If $\tan \theta = \frac{b}{a}$, show, by Demoivre's Theorem, that

$$(a - b\sqrt{-1})^{\frac{m}{n}} + (a - b\sqrt{-1})^{\frac{m}{n}} = 2(a^2 + b^2)^{\frac{m}{2n}} \cos \frac{m}{n}\theta.$$

CHAPTER XVIII.

LOGARITHMIC AND TRIGONOMETRIC SERIES.

170. *Exponential Theorem.—To show that*

$$a^x = 1 + Ax + \frac{A^2}{\underline{|2}} + \frac{A^3}{\underline{|3}}x^3 + \frac{A^4}{\underline{|4}}x^4 + \ldots\ldots$$

when $A = (a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \ldots$

Let $a-1 = a'$, then

$$a^x = (1+a')^x$$

$$= 1 + xa' + \frac{x(x-1)}{\underline{|2}}a^2 + \frac{x(x-1)(x-2)}{\underline{|3}}a^3 + \ldots \quad (1).$$

Arranging the terms of this development according to the powers of x, suppose

$$a^x = 1 + Ax + Bx^2 + Cx^3 + \ldots\ldots\ldots\ldots \quad (2).$$

Then, since x is any quantity,

$$a^y = 1 + Ay + By^2 + Cy^3 + \ldots\ldots\ldots\ldots \quad (3),$$

and $a^{x+y} = 1 + A(x+y) + B(x+y)^2 + C(x+y)^3 + \ldots$ (4).

But $a^x \times a^y = a^{x+y}$. Therefore the series (4) is equal to the product of (2) and (3); and hence, by the principle of indeterminate coefficients, the coefficient of y in (4) is equal to the coefficient of y in the product of (2) and (3). Therefore

$$A + 2Bx + 3Cx^2 + 4Dx^3 + \ldots = A + A^2x + ABx^2 + ACx^3 + \ldots$$

Applying a second time the principle of indeterminate coefficients,

$$2B = A^2, \quad 3C = A \,.\, B, \quad 4D = A \,.\, C, \quad \&c.;$$

$$\therefore\ B = \frac{A^2}{\underline{|2}}, \quad C = \frac{A \,.\, B}{3} = \frac{A^3}{\underline{|3}}, \quad D = \frac{A \,.\, C}{4} = \frac{A^4}{\underline{|4}}, \ \&c.$$

Substituting these values in (2), we have

$$a^x = 1 + Ax + \frac{A^2}{\underline{|2}}x^2 + \frac{A^3}{\underline{|3}}x^3 + \frac{A^4}{\underline{|4}}x^4 + \ldots\ldots,$$

a known series, in which, from (1),

$$A = (a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \ldots\ldots$$

171. *Logarithmic Series.*—By the exponential theorem,

$$a^x = 1 + Ax + \frac{A^2}{\underline{|2}}x^2 + \frac{A^3}{\underline{|3}}x^3 + \ldots\ldots$$

$$\text{when } A = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \ldots\ldots$$

The value of this series, when $x = \frac{1}{A}$, is usually denoted by e; hence

$$e = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \ldots\ldots\ldots\ldots \qquad (1).$$

Since $e = a^{\frac{1}{A}}$, $\therefore\ a = e^A$, $\therefore\ A = \log_e a$; and therefore

$$a^x = 1 + \log_e a \,.\, x + \frac{(\log_e a)^2}{\underline{|2}}x^2 + \frac{(\log_e a)^3}{\underline{|3}}x^3 + \ldots \qquad (2).$$

When $a = e$, $\log_e a = 1$; so that

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots\ldots\ldots\ldots \qquad (3).$$

Since $\log_e a = A$, therefore

$$\log_e a = (a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \ldots\ldots$$

Let $a = 1+x$, then $a-1 = x$, and

$$\log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots \qquad (4).$$

Replacing x by $-x$, we have

$$\log_e (1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots \qquad (5).$$

Then since $\log_e (1+x) - \log_e (1-x) = \log \frac{1+x}{1-x}$, therefore

$$\log_e \frac{1+x}{1-x} = 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\right\} \qquad (6).$$

Suppose $\frac{1+x}{1-x} = \frac{m}{n}$, then $x = \frac{m-n}{m+n}$, therefore

$$\log_e \frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \ldots\right\} (7).$$

Again, suppose $m = n+1$, then $\frac{m-n}{m+n} = \frac{1}{2n+1}$; therefore

$$\log_e \frac{n+1}{n} = 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\right\};$$

$$\therefore \log_e (n+1) =$$

$$2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\right\} + \log n \ (8).$$

172. All the series numbered in last article are worthy of notice. By means of (7) and (8) we may obtain a series of logarithms of numbers to base e; for (7) will give log 2, and (8) will then give successively logarithms for all other numbers. These logarithms would belong to Napier's system, whose base is e. Having obtained Napier's logarithms, it has been shown in Art. 137, how to obtain logarithms to any other base.

The value of e itself may be calculated from (1), and will be found to be 2·7182818.....

173. *Exponential values of the sine, cosine, and tangent.*

By the exponential theorem, e being the base of Napier's logarithms,

$$e^{x\sqrt{-1}}=1+\sqrt{-1}\,x-\frac{x^2}{\underline{|2}}-\sqrt{-1}\,\frac{x^3}{\underline{|3}}+\frac{x^4}{\underline{|4}}+\sqrt{-1}\,\frac{x^5}{\underline{|5}}-\ldots$$

and

$$e^{-x\sqrt{-1}}=1-\sqrt{-1}\,x-\frac{x^2}{\underline{|2}}+\sqrt{-1}\,\frac{x^3}{\underline{|3}}+\frac{x^4}{\underline{|4}}-\sqrt{-1}\,\frac{x^5}{\underline{|5}}-\ldots$$

Therefore, by addition,

$$e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}=2\left\{1-\frac{x^2}{\underline{|2}}+\frac{x^4}{\underline{|4}}-\ldots\ldots\right\}=2\cos x;$$

and by subtraction,

$$e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}=2\sqrt{-1}\left\{x-\frac{x^3}{\underline{|3}}+\frac{x^5}{\underline{|5}}-\ldots\right\}$$

$$=2\sqrt{-1}\sin x.$$

Hence

$$\cos x=\frac{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{2},\ \sin x=\frac{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{2\sqrt{-1}};$$

$$\therefore \tan x=\frac{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{\sqrt{-1}\left\{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}\right\}}.$$

174. *Gregory's Series, in which any angle is expanded in terms of powers of its tangent.*

$$\text{Since } \sqrt{-1}\tan a=\frac{e^{a\sqrt{-1}}-e^{-a\sqrt{-1}}}{e^{a\sqrt{-1}}+e^{-a\sqrt{-1}}},$$

therefore, by theory of proportion,

$$\frac{1+\sqrt{-1}\tan a}{1-\sqrt{-1}\tan a}=\frac{e^{a\sqrt{-1}}}{e^{-a\sqrt{-1}}}=e^{2a\sqrt{-1}};$$

$$\text{wherefore}\quad 2a\sqrt{-1}=\log_e\frac{1+\sqrt{-1}\tan a}{1-\sqrt{-1}\tan a}$$

$$=2\sqrt{-1}\left\{\tan a-\frac{1}{3}\tan^3 a+\frac{1}{5}\tan^5 a-\ldots\right\},$$

$$\text{and } a = \tan a - \frac{1}{3}\tan^3 a + \frac{1}{5}\tan^5 a - \ldots\ldots$$

This series is useless if $\tan a$ exceed unity.

175. *Euler's Series for calculating the value of* π.

$$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}} = \tan^{-1} 1 = \frac{\pi}{4}.$$

Therefore, substituting Gregory's series for $\tan^{-1}\frac{1}{2}$ and $\tan^{-1}\frac{1}{3}$,

$$\frac{\pi}{4} = \frac{1}{2} - \frac{1}{3\,.\,2^3} + \frac{1}{5\,.\,2^5} - \ldots + \frac{1}{3} - \frac{1}{3\,.\,3^3} + \frac{1}{5\,.\,3^5} - \ldots;$$

whence π is known.

176. *Machin's Series.*

Let $a = \tan^{-1}\frac{1}{5}$, that is, $\tan a = \frac{1}{5}$; then

$$\tan 2a = \frac{\frac{2}{5}}{1-\frac{1}{25}} = \frac{5}{12},$$

$$\therefore\ \tan 4a = \frac{\frac{5}{12}}{1-\frac{25}{144}} = \frac{120}{119};$$

$$\therefore\ \tan\left(4a - \frac{\pi}{4}\right) = \frac{\frac{120}{119}-1}{1+\frac{120}{119}} = \frac{1}{239};$$

$$\therefore\ 4\tan^{-1}\frac{1}{5} - \frac{\pi}{4} = \tan^{-1}\frac{1}{239},$$

or
$$\frac{\pi}{4} = 4\left\{\frac{1}{5} - \frac{1}{3\,.\,5^3} + \frac{1}{5\,.\,5^5} - \frac{1}{7\,.\,5^7} + \ldots\ldots\right\}$$
$$\left\{-\frac{1}{239} - \frac{1}{3\,(239)^3} + \frac{1}{5\,(239)^5} - \ldots\ldots\right\},$$

a rapidly converging series, by means of which it is easily determined that

$$\pi = 3{\cdot}14159265\ \ldots.$$

177. The summation of trigonometrical series will

now be exemplified. The results of the two following Articles are of some considerable importance. Those which follow them are added chiefly in order to show how examples of summation of series are usually to be treated.

178. *To find the sum of a series of sines of angles which are in arithmetical progression.*

Let the angles be α, $\alpha+\beta$, $\alpha+2\beta$, &c., and be n in number; then, writing S for the required sum,

$$S = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots + \sin\{\alpha+(n-1)\beta\}.$$

Since
$$\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{\beta}{2}\right) = 2\sin\alpha\sin\frac{\beta}{2},$$
$$\cos\left(\alpha+\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{3\beta}{2}\right) = 2\sin(\alpha+\beta)\sin\frac{\beta}{2},$$
$$\cos\left(\alpha+\frac{3\beta}{2}\right)-\cos\left(\alpha+\frac{5\beta}{2}\right) = 2\sin(\alpha+2\beta)\sin\frac{\beta}{2}$$
$$\ldots\ldots\ldots\ldots = \ldots\ldots\ldots$$
$$\cos\left\{\alpha+\left(n-\frac{3}{2}\right)\beta\right\}-\cos\left\{\alpha+\left(n-\frac{1}{2}\right)\beta\right\} = 2\sin\{\alpha+(n-1)\beta\}\sin\frac{\beta}{2}.$$

Adding,
$$\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left\{\alpha+\left(n-\frac{1}{2}\right)\beta\right\} = 2S\sin\frac{\beta}{2};$$
$$\therefore\ S = \frac{\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left\{\alpha+\left(n-\frac{1}{2}\right)\beta\right\}}{2\sin\frac{\beta}{2}},$$
or
$$S = \frac{\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

179. In the same way it is possible to find the sum of a series of cosines of angles which are in arithmetical progression; which series is also included among several whose sums may be deduced from the preceding result by the expedient of substitution.

In the series which has been already summed, put $\frac{\pi}{2}+\alpha$ for α, and it becomes

$$\sin\left(\frac{\pi}{2}+\alpha\right)+\sin\left(\frac{\pi}{2}+(\alpha+\beta)\right)+\sin\left(\frac{\pi}{2}+(\alpha+2\beta)\right)+\ldots$$

or $$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots\ldots$$

Therefore, if S be the sum of this latter series,

$$S=\frac{\sin\left(\frac{\pi}{2}+\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

$$=\frac{\cos\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

Again, let $\beta=\alpha$ in each of the series already summed. The first becomes

$$\sin\alpha+\sin 2\alpha+\sin 3\alpha+\ldots\ldots;$$

and S being its sum,

$$S=\frac{\sin\left(\alpha+\frac{n-1}{2}\alpha\right)\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}=\frac{\sin\frac{n+1}{2}\alpha\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}.$$

The second becomes

$$\cos\alpha+\cos 2\alpha+\cos 3\alpha+\ldots\ldots;$$

and writing S for its sum,

$$S=\frac{\cos\left(\alpha+\frac{n-1}{2}\alpha\right)\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}=\frac{\cos\frac{n+1}{2}\alpha\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}.$$

Further, in the series

$$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots+\sin\alpha+\{(n-1)\beta\},$$

substitute $\beta+\pi$ for β, and it becomes

$$\sin\alpha-\sin(\alpha+\beta)+\sin(\alpha+2\beta)-\ldots\ldots$$
$$+(-1)^{n-1}\sin\{\alpha+(n-1)\beta\};$$

the sum of which series therefore is

$$\frac{\sin\left\{\alpha+\frac{n-1}{2}(\beta+\pi)\right\}\sin\frac{n}{2}(\beta+\pi)}{\sin\frac{\beta+\pi}{2}}.$$

Also, performing the same substitution in the series

$$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots+\cos\{\alpha+(n-1)\beta\}$$

we have,

$$\cos\alpha-\cos(\alpha+\beta)-\cos(\alpha+2\beta)-\ldots\ldots$$
$$(-1)^{n-1}\cos\{\alpha+(n-1)\beta\};$$

the sum of which series therefore is

$$\frac{\cos\left\{\alpha+\frac{n-1}{2}(\beta+\pi)\right\}\sin\frac{n}{2}(\beta+\pi)}{\sin\frac{\beta+\pi}{2}}.$$

180. (1.) Prove that $\tan\alpha+2\tan 2\alpha+2^2\tan 2^2\alpha+\ldots$ to n terms $=\cot\alpha-2^n\cot 2^n\alpha$.

The first step here, and frequently in other cases, is suggested by putting $n=1$ in the equation to be proved.

$$\begin{aligned}\tan\alpha &= \cot\alpha-2\cot 2\alpha,\\ 2\tan 2\alpha &= 2\cot 2\alpha-2\cot^2 2^2\alpha,\\ 2^2\tan 2^2\alpha &= 2^2\cot 2^2\alpha-2^3\cot 2^3\alpha,\\ \ldots\ldots\ldots &= \ldots\ldots\ldots\\ 2^{n-1}\tan 2^{n-1}\alpha &= 2^{n-1}\cot 2^{n-1}\alpha-2^n\cot 2^n-\alpha;\end{aligned}$$

therefore, by addition,

$$\tan\alpha+2\tan 2\alpha+2^2\tan 2^2\alpha+\ldots\ldots\text{to } n \text{ terms} = \cot\alpha-2^n\cot 2^n\alpha.$$

(2.) Sum to n terms the series

$$\sin^3\alpha+\sin^3(\alpha+\beta)+\sin^3(\alpha+2\beta)+\ldots\ldots$$

Let S be the required sum. Since

$$3 \sin \alpha - 4 \sin^3 \alpha = \sin 3\alpha,$$
$$\therefore\ 4 \sin^3 \alpha = 3 \sin \alpha - \sin 3\alpha.$$

Similarly,
$$4 \sin^3 (\alpha+\beta) = 3 \sin (\alpha+\beta) - \sin 3 (\alpha+\beta),$$
$$4 \sin^3 (\alpha+2\beta) = 3 \sin (\alpha+2\beta) - \sin 3 (\alpha+2\beta),$$
$$\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots,$$

and $4 \sin^3 [\alpha+(n-1)\beta] = 3 \sin [\alpha+(n-1)\beta] - \sin 3 [\alpha+(n-1)\beta]$.
Adding,

$$4S = 3\left\{\sin \alpha + \sin (\alpha+\beta) + \ldots\ldots + \sin [\alpha+(n-1)\beta]\right\}$$
$$- \left\{\sin 3\alpha + \sin 3 (\alpha+\beta) + \ldots\ldots + \sin 3 [\alpha+(n-1)\beta]\right\}.$$

Applying the result of Art. 178,

$$S = \frac{3}{4} \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right) \sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} - \frac{1}{3} \frac{\sin \left\{3\alpha + \frac{3(n-1)}{2}\beta\right\} \sin \frac{3n\beta}{2}}{\sin \frac{3\beta}{2}}.$$

(3.) Prove, by means of the exponential values of the functions involved, that

$$\sin 2\theta = 2 \sin \theta \cos \theta.$$

Writing i for $\sqrt{-1}$,

$$2 \sin \theta \cos \theta = 2 \frac{e^{\theta i} + e^{-\theta i}}{2} \times \frac{e^{\theta i} - e^{-\theta i}}{2\sqrt{-1}}$$
$$= \frac{e^{2\theta i} - e^{-2\theta i}}{2\sqrt{-1}} = \sin 2\theta.$$

(4.) Sum the series

$$1 + x \cos \theta + x^2 \cos 2\theta + x^3 \cos 3\theta + \ldots\ldots + x^{n-1} \cos (n-1)\theta.$$

Writing i for $\sqrt{-1}$,

$$1 + x \cos \theta + x^2 \cos 2\theta + x^3 \cos 3\theta + \ldots\ldots + x^{n-1} \cos (n-1)\theta$$
$$= 1 + x \frac{e^{\theta i} + e^{-\theta i}}{2} + x^2 \frac{e^{2\theta i} + e^{-2\theta i}}{2} + \ldots + x^{n-1} \frac{e^{(n-1)\theta i} + e^{-(n-1)\theta i}}{2}$$
$$= \frac{1}{2}\left\{1 + xe^{\theta i} + x^2 e^{2\theta i} + \ldots\ldots \text{ to } n \text{ terms}\right\}$$
$$+ \frac{1}{2}\left\{1 + xe^{-\theta i} + x^2 e^{-2\theta i} + \ldots\ldots \text{ to } n \text{ terms}\right\}$$
$$= \frac{1}{2} \frac{x^n e^{n\theta i} - 1}{xe^{\theta i} - 1} + \frac{1}{2} \frac{x^n e^{-n\theta i} - 1}{xe^{-\theta i} - 1}$$

$$= \frac{1}{2} \cdot \frac{(x^n e^{n\theta i}-1)\,(xe^{-\theta i}-1) + (x^n e^{-n\theta i}-1)\,(xe^{\theta i}-1)}{(xe^{\theta i}-1)\,(xe^{-\theta i}-1)}$$

$$= \frac{1}{2} \cdot \left\{ \frac{x^{n+1} e^{(n-1)\theta i} - x^n e^{n\theta i} - xe^{-\theta i} + 1}{x^2 - xe^{\theta i} - xe^{-\theta i} + 1} \right.$$

$$\left. + \frac{x^{n+1} e^{-(n-1)\theta i} - x^n e^{-n\theta i} - xe^{\theta i} + 1}{x^2 - xe^{\theta i} - xe^{-\theta i} + 1} \right\}$$

$$= \frac{1}{2} \cdot \frac{x^{n+1}(e^{(n-1)\theta i} + e^{-(n-1)\theta i}) - x^n(e^{n\theta i} + e^{-n\theta i}) - x(e^{\theta i} + e^{-\theta i}) + 2}{x^2 - x\,(e^{\theta i} + e^{-\theta i}) + 1}$$

$$= \frac{1}{2} \cdot \frac{2x^{n+1} \cos (n-1)\,\theta - x^n . 2 \cos n\theta - x . 2 \cos \theta + 2}{x^2 - 2x \cos \theta + 1}$$

$$= \frac{x^{n+1} \cos (n-1)\,\theta - x^n \cos n\theta - x \cos \theta + 1}{x^2 - 2x \cos \theta + 1}.$$

(5.) Sum the infinite series

$$c \sin (\alpha+\beta) + \frac{c^2}{1.2} (\sin \alpha + 2\beta) + \frac{c^3}{1.2.3} \sin (\alpha + 3\beta) + \ldots\ldots$$

Write S for the required sum, and i for $\sqrt{-1}$. Substituting the exponential values of the sines, we have

$$2 . iS = ce^{(\alpha+\beta)i} + \frac{c^2}{1.2} e^{-(\alpha+2\beta)i} + \frac{c^3}{1.2.3} e^{(\alpha+3\beta)i} + \ldots\ldots$$

$$- ce^{-(\alpha+\beta)i} - \frac{c^2}{1.2} e^{-(\alpha+2\beta)i} - \frac{c^3}{\underline{|3}} e^{-(\alpha+3\beta)i} - \ldots\ldots$$

$$= e^{\alpha i}\left\{ e^{ce^{\beta i}} - 1 \right\} - e^{-\alpha i}\left\{ e^{ce^{-\beta i}} - 1 \right\}.$$

But $e^{\beta i} = \cos \beta + i \sin \beta$, $e^{-\beta i} = \cos \beta - i \sin \beta$;

therefore

$$2iS = e^{c \cos \beta + i\,(\alpha + c \sin \beta)} - e^{\cos \beta - i\,(\alpha + c \sin \beta)} - (e^{\alpha i} - e^{-\alpha i})$$

$$= e^{c \cos \beta}\left\{ e^{i\,(\alpha + c \sin \beta)} - e^{-i\,(\alpha + c \sin \beta)} \right\} - 2i \sin \alpha;$$

$$\therefore\ S = e^{c \cos \beta} \sin (\alpha + c \sin \beta) - \sin \alpha.$$

(6.) Sum to n terms the series

$$\tan^{-1} \frac{1}{1+1+1^2} + \tan^{-1} \frac{1}{1+2+2^2} + \tan^{-1} \frac{1}{1+3+3^2} + \ldots\ldots$$

Let $\tan^{-1}\dfrac{1}{1+n+n^2} = \alpha-\beta$; then

$$\frac{1}{1+n+n^2} = \tan(\alpha-\beta)$$

or $$\frac{(1+n)-n}{1+(1+n)n} = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta};$$

from which it follows that

$$\tan\alpha = n+1, \quad \therefore\ \alpha = \tan^{-1}(n+1);$$

$$\text{and } \tan\beta = n, \quad \therefore\ \beta = \tan^{-1} n.$$

Hence $\tan^{-1}\dfrac{1}{1+n+n^2} = \tan^{-1}(n+1)-\tan^{-1} n.$

Wherefore, substituting 1, 2, 3, &c., for n,

$$\tan^{-1}\frac{1}{1+1+1^2} = \tan^{-1} 2 - \tan^{-1} 1,$$

$$\tan^{-1}\frac{1}{1+2+2^2} = \tan^{-1} 3 - \tan^{-1} 2,$$

$$\tan^{-1}\frac{1}{1+3+3^2} = \tan^{-1} 4 - \tan^{-1} 3,$$

$$\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$$

$$\text{and } \tan^{-1}\frac{1}{1+n+n^2} = \tan^{-1}(n+1)-\tan^{-1} n.$$

Therefore, by addition,

$$\tan^{-1}\frac{1}{1+1+1^2} + \tan^{-1}\frac{1}{1+2+2^2} + \tan^{-1}\frac{1}{1+3+3^3} + \ldots \text{ to } n \text{ terms}$$

$$= \tan^{-1}(n+1)-\tan^{-1} 1$$

$$= \tan^{-1}(n+1)-\frac{\pi}{4}.$$

Examples.

Apply the method of Art. 178 to obtain the sum of n terms of the following series :—

1. $\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\cos(\alpha+3\beta)+\ldots\ldots$
2. $\sin\alpha-\sin(\alpha+\beta)+\sin(\alpha+2\beta)-\sin(\alpha+3\beta)+\ldots\ldots$
3. $\cos\alpha-\cos(\alpha+\beta)+\cos(\alpha+2\beta)-\cos(\alpha+3\beta)+\ldots\ldots$
4. $\sin\alpha+\sin 2\alpha+\sin 3\alpha+\ldots\ldots$
5. $\cos\alpha+\cos 2\alpha+\cos 3\alpha+\ldots\ldots$
6. $\operatorname{cosec}\theta+\operatorname{cosec} 2\theta+\operatorname{cosec} 2^2\theta+\ldots\ldots$

7. $\tan\theta + \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{2^2}\tan\frac{\theta}{2^2} + \ldots\ldots$

8. $\sec\theta\sec 2\theta + \sec 2\theta\sec 3\theta + \ldots\ldots$

9. $(\tan x + \cot x) + (\tan 2x + \cot 2x) + (\tan 4x + \cot 4x) + \ldots\ldots$

10. Sum the series

$$\cos 1^\circ + \cos 3^\circ + \cos 5^\circ + \cos 7^\circ + \ldots\ldots \text{ to 15 terms.}$$

Prove, by means of the exponential values of the functions involved, that

11. $\sin^2\theta = 1 - \cos^2\theta.$

12. $\cos 2a = \cos^2 a - \sin^2 a.$

13. $\tan 2a = \dfrac{2\tan a}{1-\tan^2 a}.$

14. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$

Sum to n terms the series :—

15. $\cos^2\theta + \cos^2 2\theta + \cos^2 3\theta + \ldots\ldots$

16. $\sin^2\theta + \sin^2 2\theta + \sin^2 3\theta + \ldots\ldots$

17. $\sin^3\theta + \sin^3 2\theta + \sin^3 3\theta + \ldots\ldots$

18. $\cos^3 a + \cos^3(a+\beta) + \cos^3(a+2\beta) + \ldots\ldots$

19. $\tan^{-1}x + \tan^{-1}\dfrac{x}{1+1\,.\,2x^2} + \tan^{-1}\dfrac{x}{1+2\,.\,3x^2} + \ldots\ldots$

20. $x\sin\theta + x^2\sin 2\theta + x^3\sin 3\theta + \ldots\ldots$

21. $\sin a + x\sin(a+\beta) + x^2\sin(a+2\beta) + \ldots\ldots$

Sum to infinity the series :—

22. $\frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{4}\tan\frac{\theta}{4} + \frac{1}{8}\tan\frac{\theta}{8} + \ldots\ldots$

23. $\cos\theta + \dfrac{\cos\theta}{1}\cos 2\theta + \dfrac{\cos^2\theta}{\underline{|2}}\cos 3\theta + \dfrac{\cos^3\theta}{\underline{|3}}\cos 4\theta + \ldots\ldots$

24. $x\sin\theta + \dfrac{x^2}{2}\sin 2\theta + \dfrac{x^3}{3}\sin 3\theta + \ldots\ldots$

HINTS FOR SOLUTION OF SOME OF THE EXAMPLES.

CHAPTER I.

Ex. 9. Apply Euc. I. 32, Cor 1.

CHAPTER III.

Ex. 32. Transpose, and add 1 to each side; then

$$\tan^2\theta - 2\tan\theta + 1 = \tan\theta + 4\sqrt{\tan\theta} + 4;$$

each side of which is a perfect square.

Ex. 39. Apply the formula $\sin^2\theta + \cos^2\theta = 1$.

CHAPTER V.

Ex. 10. Solve by means of *two* right-angled triangles, including in the figure a line from first position of ship due eastward.

CHAPTER VI.

Ex. 2. Apply (2) in Art. 50.

Ex. 8, 9. Prove after the manner of Art. 43.

CHAPTER VIII.

Ex. 7. When the least angle which satisfies the equation is $0, \frac{\pi}{2}, \pi$, or $\frac{3\pi}{2}$, a simpler general form for the root should be sought than that which an application of the formulæ would give.

CHAPTER IX.

Ex. 19. The angles $90° + A$ and $90° - A$ are supplements.

Ex. 29. $\cos^6\alpha - \sin^6\alpha = (\cos^2\alpha - \sin^2\alpha)(\cos^4\alpha + \cos^2\alpha\sin^2\alpha + \sin^4\alpha)$.

Ex. 43. In formula $\sin(A+B) = \sin A \cos B + \cos A \sin B$, put $-B$ for B, and apply $\sin(-B) = -\sin B$, $\cos(-B) = \cos B$.

Ex. 44. $\sin(A-B) = \cos\left\{90^\circ - (A-B)\right\} = \cos\left\{A - (90^\circ + B)\right\}$.

CHAPTER X.

Ex. 26. $4\cos A \cos(120^\circ - A)\cos(120^\circ + A)$

$$= 2\cos A \,.\, 2\cos(120^\circ + A)\cos(120^\circ - A).$$

Ex. 40—53. The answers to these equations are given in a form which affords some hint of the mode of solution in each case.

Ex. 46. Multiply both sides by $\frac{1}{\sqrt{2}}$.

Ex. 50. The angles $\frac{\pi}{4} + \alpha$ and $\frac{\pi}{4} - \alpha$ are complements.

Ex. 63. Apply (1) $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$.

(2) $\tan A + \tan B = \dfrac{\sin(A+B)}{\cos A \cos B} = \dfrac{2\sin(A+B)}{\cos(A+B) + \cos(A-B)}$.

CHAPTER XI.

Ex. 1—6. The formula $\frac{\sin A}{a} = \frac{\sin B}{b}$ should be familiarly known under the two other forms

$$\frac{\sin A}{\sin B} = \frac{a}{b},\ b\sin A = a\sin B.$$

Ex. 11. $\cos(B+C) = -\cos A$.

ANSWERS TO THE EXERCISES.

EXERCISE 1.

1. $\frac{2}{5}$ or 2 : 5, 2 or 2 : 1, $\frac{2}{3}$ or 2 : 3, $\frac{3}{13}$ or 3 : 13, $\frac{1}{3}$ or 1 : 3, $\frac{3}{17}$ or 3 : 17.

2. 2 : 3, 24 : 7, 12 : 1, 9 : 16.

3. The first ratio is three times the second.

4. The ratios are equal.

EXERCISE 3.

1. 585°. **2.** 1170°.

4. 50° is in the first, 250° in third, $-450°$ is in no quadrant but between third and fourth, 650° is in the fourth, $-850°$ in third, $-1000°$ in first, $\frac{5\pi}{3}$ in fourth, $2n\pi + \frac{5\pi}{6}$ in second, $\frac{-4\pi}{3}$ in third, and $-\left\{(2n+1)\pi + \frac{2\pi}{3}\right\}$ in first quadrant.

5. Putting A for angle AOP_2,

$$\sin A = \frac{4}{5}, \quad \cos A = -\frac{3}{5}, \quad \tan A = -\frac{4}{3}, \quad \operatorname{cosec} A = \frac{5}{4},$$

$$\sec A = -\frac{5}{3}, \quad \cot A = -\frac{3}{4}.$$

6. If A be the angle AOP_3,

$$\sin A = \frac{-y}{r}, \quad \cos A = \frac{-x}{r}, \quad \tan A = \frac{y}{x}, \quad \cot A = \frac{x}{y},$$

$$\sec A = -\frac{r}{x}, \quad \operatorname{cosec} A = -\frac{r}{y}.$$

EXERCISE 5.

2. $2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$, $\cos^2 \frac{\theta}{4} - \sin^2 \frac{\theta}{4}$, $\dfrac{2 \tan \frac{\theta}{6}}{1 - \tan^2 \frac{\theta}{6}}$.

3. $1-2\sin^2\frac{5A}{2}$, $\frac{2\tan 3A}{1-\tan^2 3A}$.

5. $\frac{1}{2}\sqrt{2+\sqrt{2}}$. **6.** $\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$.

7. $\frac{\cot A\cot B-1}{\cot B+\cot A}$, $\frac{\cot A\cot B+1}{\cot B-\cot A}$, $\frac{\cot^2 A-1}{2\cot A}$.

8. $2+\sqrt{3}$.

EXERCISE 7.

3. $\log x+\log y+\log z$. **4.** $1+\log 2$.

5. $\log a+2\log b+3\log c-2\log 2-4\log x-5\log y$.

6. $-2\log 2$. **7.** $2+\log 3-\log 2$.

8. $\log 2+\log 3+\log a+\frac{3}{2}\log b+\frac{1}{2}\log c+\frac{1}{2}\log d$.

9. $\log 2-\frac{15}{4}\log 3$. **10.** $\frac{13}{20}\log 2+\frac{2}{3}\log 3-\cdot 25$.

11. $\log c+\log\sin B-\log\sin C$. **12.** $\log a+\log c+\log\sin B-2$.

13. $\frac{1}{2}\left\{\log(b+c)+\log(b-c)\right\}$.

14. $\frac{1}{2}\left\{\log(s-b)+\log(s-c)-\log b-\log c\right\}$.

15. $\log(a-b)+\log\cot\frac{C}{2}-\log(a+b)$.

16. $\log(a+b)+\log\sin\frac{C}{2}+\log\sec\phi$.

ANSWERS.

CHAPTER I.

1. $45°$, $67° 30'$, $139° 5' 27''\cdot\dot{2}\dot{7}$, $133^g 33^c 33^m\cdot\dot{3}$, $85^g 17^c 42^m\cdot 86$.

2. $\frac{\pi}{4}$ or $\cdot 7854$, $\frac{3\pi}{8}$ or $1\cdot 1781$, $\frac{3\pi}{7}$ or $1\cdot 3464$, $\frac{2\pi}{3}$ or $2\cdot 0944$, $\frac{17\pi}{22}$ or $2\cdot 4276$.

3. $\frac{\pi}{6}$ or $\cdot 5236$, $\frac{\pi}{8}$ or $\cdot 3927$, $2\cdot 7358$.

4. $\frac{3\pi}{4}$ or $2\cdot 3562$, $1\cdot 1564$.

5. $150°$, $30° 30'$, $48° 24' 12''$.

6. 40^g, $111^g 27^c 16^m$ nearly.

7. $166^g 66^c 66^m\cdot 7$, $20^g 13^c 40^m$.

8. $112° 30'$, $50° 37' 0''\cdot 3$.

9. $60°$, $120°$, $144°$, $156°$; $\frac{\pi}{3}$, $\frac{2\pi}{3}$, $\frac{4\pi}{5}$, $\frac{13\pi}{15}$.

10. $2\cdot\dot{6}$.

11. $\cdot\dot{6}$.

CHAPTER II.

6. $\dfrac{\tan A}{\sqrt{1+\tan^2 A}}$, $\dfrac{\sqrt{\operatorname{cosec}^2 A - 1}}{\operatorname{cosec} A}$.

7. $\dfrac{\sqrt{1-\cos^2 A}}{\cos A}$, $\dfrac{1}{\sqrt{\sec^2 A - 1}}$.

8. $\dfrac{1}{\sqrt{1-\sin^2 A}}$, $\dfrac{1}{\sqrt{\operatorname{vers} A\,(2-\operatorname{vers} A)}}$.

9. $\sin A = \frac{3}{5}$, $\cos A = \frac{4}{5}$, $\cot A = \frac{4}{3}$, $\sec A = \frac{5}{4}$, $\operatorname{cosec} A = \frac{5}{3}$, $\operatorname{vers} A = \frac{1}{5}$.

10. $\cos A = \frac{\sqrt{3}}{2}$, $\tan A = \frac{1}{\sqrt{3}}$, $\operatorname{vers} A = \frac{2-\sqrt{3}}{2}$.

11. $\sin A = \frac{1}{2}\sqrt{\left(\frac{7}{2}\right)} = \frac{1}{4}\sqrt{14}$. $\cot A = \frac{1}{\sqrt{7}}$.

CHAPTER III.

1. 78°, 64° 56′, 33° 9′ 16″. 1° 1′ 1″·11.

7. 45°. **8.** 15°. **9.** 18°.

10. Secants are $\frac{2}{\sqrt{3}}$, $\sqrt{2}$, 2; cosecants 2, $\sqrt{2}$, $\frac{2}{\sqrt{3}}$; cotangents are $\sqrt{3}$, 1, $\frac{1}{\sqrt{3}}$; and versed sines $\frac{2-\sqrt{3}}{2}$, $\frac{\sqrt{2}-1}{\sqrt{2}}$, $\frac{1}{2}$.

18. 45°. **19.** 30°. **20.** 45°. **21.** 60°.

22. 45°. **23.** 30°. **24.** 60°. **25.** 60°.

26. 60°. **27.** 30°. **28.** $\pm\frac{1}{2}$. **29.** $\pm\frac{2}{3}$.

30. $\frac{1}{3}$ or -1. **31.** $2\pm\sqrt{3}$. **32.** $\frac{1}{2}(7\pm\sqrt{13})$. **33.** 45°, 30°.

34. 45°, 15°. **35.** 45°, 45°. **36.** 30°, 60°. **37.** $\pm\frac{n}{\sqrt{m^2+n^2}}$.

38. $\frac{m+n}{m-n}$.

CHAPTER IV.

1. 2·5. **2.** 10. **3.** $20\sqrt{2}$.

4. 30°. **5.** 60°. **6.** 45°.

7. $B=60°$, $a=25$, $b=43·3$. **8.** $A=60°$, $b=1000$, $c=2000$.

9. $A=30°$, $B=60°$, $c=100$. **10.** $A=45°$, $B=45°$, $a=36$.

11. 14·5. **12.** ·8192. **13.** 51° 41′. **14.** 33° 34′.

15. $BC=500$, $CA=866$, $AB=1000$. **16.** $PH=PQ=100$.

17. $\frac{b\tan\alpha}{\tan\beta}$. **18.** $\frac{a}{\tan\alpha-\tan\beta}$. **19.** 15°. **20.** $a\sqrt{3}$.

CHAPTER V.

1. $15\sqrt{3}$ feet. **2.** $\frac{100\sqrt{3}}{3}$ yards.

3. $6(\sqrt{6}-\sqrt{2})$ miles, $12\sqrt{2}$ miles. **4.** $\sqrt{3}:1$.

5. $15(3+\sqrt{3})$ feet. **7.** $\sqrt{3}$ miles, $1\frac{1}{2}$ miles.

8. 42 feet, 18 feet. **9.** $\sqrt{3}:1$.

10. $14\sqrt{2}$ miles, $7(\sqrt{6}+\sqrt{2})$ miles.

11. $8(4-\sqrt{2})$ miles, $4(3\sqrt{3}-\sqrt{6}+\sqrt{2})$ miles.

12. $1041\frac{2}{3}$ feet. **13.** 117·56 yards.

14. 11·547 feet, 23·094 feet. **15.** 6·3 miles.

16. 108·64 yards. **17.** 629·8 yards.

18. $c = \dfrac{a+b}{\sin A + \cos A}$, $a = \dfrac{(a+b)\sin A}{\sin A + \cos A}$, $b = \dfrac{(a+b)\cos A}{\sin A + \cos A}$.

19. $c = \dfrac{a-b}{\sin A - \cos A}$, $a = \dfrac{(a-b)\sin A}{\sin A - \cos A}$, $b = \dfrac{(a-b)\cos A}{\sin A - \cos A}$.

20. $a = \dfrac{(b-c)\sin A}{\cos A - 1}$.

21. $c = \dfrac{a+b+c}{1+\sin A+\cos A}$, $a = \dfrac{(a+b+c)\sin A}{1+\sin A+\cos A}$, $b = \dfrac{(a+b+c)\cos A}{1+\sin A+\cos A}$.

CHAPTER VI.

1. $-\cos 20°$, $\tan 80°$, $-\sec 20°$, $\operatorname{cosec} 80°$.

3. $-\cos \alpha$, $\cot \alpha$, $-\tan \theta$, $\sin \theta$.

4. $-\dfrac{1}{\sqrt{2}}$, 1, -2.

5. $\dfrac{\sqrt{3}}{2}$, $\dfrac{\sqrt{3}}{2}$, -1, $-\dfrac{1}{\sqrt{3}}$.

6. $150°$, $210°$, $510°$.

7. $\dfrac{\pi}{4}$, $\dfrac{3\pi}{4}$, $\dfrac{5\pi}{4}$, $\dfrac{7\pi}{4}$, $-\dfrac{\pi}{4}$, $-\dfrac{3\pi}{4}$, $-\dfrac{5\pi}{4}$, $-\dfrac{7\pi}{4}$.

10. $\sin \theta = \dfrac{1}{2}$; $\therefore\ \theta = 30°$ or $150°$.

11. $60°$ or $300°$.

12. $60°$, $120°$, $30°$, $150°$.

13. $45°$.

14. $\cos A$, $\sin A$, $-\cos A$, $-\sin A$, and these recurring.

CHAPTER VII.

5. Tan A_1 and tan A_2 are both very large, but of opposite signs. The same may be said of sec A_1 and sec A_2.

6. & 7. Values of A in second quadrant, and nearly equal to $180°$; also in fourth quadrant, and nearly equal to $360°$.

9. 1 and -1.

10. Sine of cotangent but not of secant.

11. $\dfrac{\pi}{2}$, $\dfrac{3\pi}{2}$.

12. 0.

13. π, $\dfrac{\pi}{3}$, $\dfrac{5\pi}{3}$.

14. 0, π, $\dfrac{\pi}{3}$, $\dfrac{4\pi}{3}$.

15. $\dfrac{\pi}{3}$, $\dfrac{2\pi}{3}$, π.

CHAPTER VIII.

1. $n\pi + (-1)^n \dfrac{\pi}{3}$.

2. $n \,.\, 180° + 60°$ or $n \,.\, 180° + 120°$.

3. $n\pi + (-1)^n \dfrac{\pi}{4}$.

4. $2n\pi \pm \dfrac{\pi}{6}$.

5. $2n\pi \pm \frac{3\pi}{4}$.

6. $2n\pi \pm \frac{\pi}{6}$.

7. $(2n+1)\pi$.

8. $2n.180° + 90°$.

9. $(2n+1)\frac{\pi}{2}$.

10. $(2n+1)\frac{\pi}{2}$, or $(6n+5)\frac{\pi}{6}$.

CHAPTER IX.

35. $\dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}$.

36. $\dfrac{\cot A \cot B \cot C - \cot A - \cot B - \cot C}{\cot B \cot C + \cot C \cot A + \cot A \cot B - 1}$.

38. $2 - \sqrt{3}$.

39. $\dfrac{\sec A \sec B}{1 - \sqrt{(\sec^2 A - 1)(\sec^2 B - 1)}}$.

CHAPTER X.

40. $\frac{3\theta}{2} = n\pi$ or $\frac{\theta}{2} = 2n\pi \pm \frac{\pi}{2}$; that is, $\theta = n\frac{2\pi}{3}$ or $(4n \pm 1)\pi$.

41. $2\theta = n\pi$ or $\theta = 2n\pi \pm \frac{\pi}{3}$.

42. $2\theta = \left(n + \frac{1}{2}\right)\pi$, or $\theta = 2n\pi \pm \frac{2\pi}{3}$.

43. $\frac{3\theta}{4} = 2n\pi \pm \frac{\pi}{2}$, or $\theta = 2n\pi \pm \frac{2\pi}{3}$.

44. $3\theta = n\pi$, or $4\theta = 2n\pi \pm \frac{\pi}{3}$.

45. $\theta = 2n\pi \pm \frac{\pi}{2}$, or $(n-1)\theta = 2n\pi \pm \frac{\pi}{3}$.

46. $\theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3}$.

47. $\theta - \frac{\pi}{4} = 2n\pi$.

48. $\theta + \frac{\pi}{3} = 2n\pi \pm \frac{3\pi}{4}$.

49. $2\theta = n\pi + (-1)^n \frac{\pi}{6}$.

50. $\frac{\pi}{4} + a = n\pi + \frac{\pi}{3}$.

51. $a = \frac{n\pi}{8}$ or $\frac{n\pi}{4}$.

52. $3a + \frac{\pi}{2} = 2n\pi \pm a$.

53. $2a = n\pi + (-1)^n \frac{\pi}{6}$.

54. $\sin 7° = \dfrac{\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}}{4}$, $\sin 9° = \dfrac{\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}}{4}$.

55. $\sin 105° = \frac{1+\sqrt{3}}{2\sqrt{2}}$, $\cos 105° = \frac{1-\sqrt{3}}{2\sqrt{2}}$.

56. $\sin 7\frac{1}{2}° = \frac{\sqrt{2\sqrt{2}+\sqrt{3}-1}-\sqrt{2\sqrt{2}-\sqrt{3}+1}}{2\sqrt[4]{8}}$,

57. $\cos 7\frac{1}{2}° = \frac{\sqrt{2\sqrt{2}+\sqrt{3}-1}+\sqrt{2\sqrt{2}-\sqrt{3}+1}}{2\sqrt[4]{8}}$.

82. $x = \frac{1}{8}$ or -1. **83.** $x = -1$.

84. $x = \frac{a+b}{1-ab}$. **85.** $x = 0$ or $\pm\frac{1}{2}$.

CHAPTER XI.

27. 90°. **28.** 60°. **29.** 7.

30. 60°, 75°, 6 yds. **31.** 2. **32.** 60°.

33. 45° or 135°. **34.** 30°. **35.** Impossible.

36. $B = 90°$, $C = 72°$, $c = 2\sqrt{5+2\sqrt{5}}$. Angle 90° and its supplement are the same; the triangle is right-angled.

37. 90°, 36° 52′. **38.** 62° 11′. **39.** 130° 27′.

40. 10. **41.** 1 mile.

42. 48° 25′ or 131° 35′. **43.** 120°.

44. If there be an obtuse angle, it is opposite to the greatest side. The square of that side is therefore greater than the sum of the squares of the other two sides. (Euc. II. 12.)

CHAPTER XII.

27. $a = 2R \sin A$ &c. **28.** 26·83. **29.** 173·2.

30. 6. **31.** 1 : 2.

CHAPTER XIII.

1. 3, $\bar{1}$, 2, $\bar{3}$, 0, $\bar{5}$, 2, 3, 0.

2. 306, 5642, 841, 1, 11, 11001.

5. $\bar{1}$·06818, 2·06818, $\bar{4}$·06818. **6.** 8, 5·$\dot{3}$. **7.** 2, $\bar{3}$, $\bar{2}$·5.

8. 2·$\dot{8}$. **9.** $\frac{np}{mq}$. **11.** ·00001.

12. 31·622, 2·154, 21·544. **13.** 3·162.....

15. $\sqrt{10}$ or 3·162..... **16.** Multiply by $\frac{8}{3}$.

17. ·6453. **18.** ·221. **19.** 1·86.

20. 1·357. **21.** 1·06. **22.** 5·435.
23. 3·7737. **24.** 2·10721, 2·09691, 3·39794.
25. $\bar{4}$·69897, 2·2922561, ·561141.
26. $-$·150515 = $\bar{1}$·949485, ·2385607, ·0624693.
27. ·477121, $\bar{1}$·204120, 2·653212.
28. 17. **29.** 112376. **30.** 9·936095.

CHAPTER XIV.

1. 6·2369587. **2.** ·78066589. **3.** ·0209675.
4. Prop. parts for 1, 2, 3, 4, 5, 6, 7, 8, 9, are 10, 19, 29, 38, 48, 58, 67, 77, 86, respectively.
5. Prop. parts for 1, 2, 3, 4, 5, 6, 7, 8, 9, are 26, 52, 79, 105, 131, 157, 183, 210, 236, respectively.
6. $\bar{1}$·7406162, $\bar{1}$·6366007. **7.** ·02031379. **8.** 2·653375.
9. 2. **10.** $\bar{2}$·778066. **11.** 1·70279.
14. 2. **18.** ·5366616. **19.** 47° 4′ 3″.
20. 75° 56′ 49″. **21.** 9·9633198. **22.** 9·3275987.
23. 9·8684953. **24.** 53° 25′ 20″·3. **25.** 37° 40′ 14″.

CHAPTER XV.

1. 1239·632857 feet. **2.** 51° 28′ 3″.
3. 101° 29′ 9″, 36° 0′ 51″. **4.** 9·6733937.
5. 119° 26′ 51″·3, 5° 33′ 8″·7. **6.** 55° 45′ 44″.
7. 69° 10′ 10″, 46° 37′ 50″. **8.** 82° 10′ 50″, 50° 24′ 10″.
9. 48° 11′ 23″, 58° 24′ 43″, 73° 23′ 54″.
10. 117° 19′ 11″, 2° 40′ 49″. **11.** 41536·18 yards.
12. 71·0919$\dot{3}$ feet. **13.** 38° 15′ 8″, 51° 44′ 52″.
15. $3(2\sqrt{3}-1)$ feet. **16.** $\frac{3+\sqrt{3}}{2}$ miles.
17. $\frac{a^2+c^2}{a-c}$. **21.** 45° or $\tan^{-1} 2$.
22. 100 feet. **24.** $10\sqrt{115}$ feet.
27. $18\sqrt{2}$ miles. **28.** 93·489 yards.
29. 63° 26′ 6″. **30.** 155·8 feet.

31. 519·6 yards.

32. 1 ac. 4 sq. ch. 6968·$\dot{3}$ sq. lks.

33. 116° 33′ 54″, 26° 33′ 54″.

34. 618·17 feet, 212·85 feet.

35. 238·5 yards, 480·5 yards.

36. 296·40031 feet.

CHAPTER XVI.

4. 10 and 9. **6.** 171° 53′ 14″. **8.** 25 : 4 : 9. **10.** 6.

CHAPTER XVII.

1. $\cos\frac{4\alpha}{3}+\sqrt{-1}\sin\frac{4\alpha}{3}$. **2.** 1, $\frac{-1\pm\sqrt{-3}}{2}$.

3. -1, $\frac{1\pm\sqrt{-3}}{2}$.

4. $\frac{1}{32}\cos 6\theta+\frac{3}{16}\cos 4\theta+\frac{15}{32}\cos 2\theta+\frac{5}{16}$;
$-\frac{1}{32}(\cos 6\theta-6\cos 4\theta+15\cos 2\theta-10)$.

5. $\frac{1}{16}(\sin 5\theta-5\sin 3\theta+10\sin\theta)$;
$-\frac{1}{64}(\sin 7\theta-7\sin 5\theta+21\sin 3\theta-35\sin\theta)$.

6. $4\sin\theta\cos^3\theta-4\cos\theta\sin^3\theta$;
$\cos^7\theta-21\sin^2\theta\cos^5\theta+35\sin^4\theta\cos^3\theta-7\sin^6\theta\cos\theta$.

7. 1, $\frac{\sqrt{5}-1}{4}\pm\frac{\sqrt{-10-2\sqrt{5}}}{4}$, $-\frac{\sqrt{5}+1}{4}\pm\frac{\sqrt{-10+2\sqrt{5}}}{4}$.

CHAPTER XVIII.

6. $\cot\frac{\theta}{2}-\cot 2^{n-1}\theta$. **7.** $\frac{1}{2^{n-1}}\cot\frac{\theta}{2^{n-1}}-2\cot 2\theta$.

8. $\operatorname{cosec}\theta\left\{\tan(n+1)\theta-\tan\theta\right\}$.

9. $2\cot x-2\cot 2^n x$. **10.** $\frac{1}{4}\operatorname{cosec}1^\circ$.

15. $\frac{1}{4}\left\{2n-1+\sin(2n+1)\theta\operatorname{cosec}\theta\right\}$.

16. $\frac{1}{4}\left\{2n+1-\sin(2n+1)\theta\operatorname{cosec}\theta\right\}$.

17. $\frac{3\sin\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta}{4\sin\frac{1}{2}\theta}-\frac{\sin\frac{3}{2}(n+1)\theta\sin\frac{3}{2}n\theta}{4\sin\frac{3}{2}\theta}$.

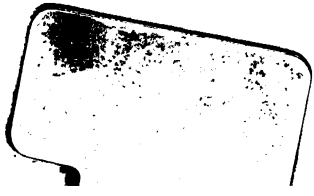
18. $\frac{3}{4}\frac{\cos\left\{\alpha+\frac{1}{2}(n-1)\beta\right\}\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}+\frac{1}{4}\frac{\cos\left\{3\alpha+\frac{1}{2}(n-1)3\beta\right\}\sin\frac{3n\beta}{2}}{\sin\frac{3\beta}{2}}$

19. $\tan^{-1}nx$.

20. $\dfrac{x^{n+2}\sin n\theta - x^{n+1}\sin(n+1)\theta + x\sin\theta}{x^2-2x\cos\theta+1}$.

21. $\dfrac{\sin\alpha - x\sin(\alpha-\beta) - x^n\sin(\alpha+n\beta) + x^{n+1}\sin\left\{\alpha+(n-1)\beta\right\}}{1-2x\cos\beta+x^2}$

22. $\dfrac{1}{\theta} - \cot\theta$.

23. $\frac{1}{2}\left(xe^{x\cos\theta} + x^{-1}e^{x^{-1}\cos\theta}\right)$.

24. $\tan^{-1}\left(\dfrac{x\sin\theta}{1-x\cos\theta}\right)$.